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Use of the Compass  
in  
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# USE OF THE COMPASS IN GEOMETRICAL CONSTRUCTION

BY

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## THESIS

FOR THE DEGREE OF MASTER OF SCIENCE

IN THE GRADUATE SCHOOL

UNIVERSITY OF ILLINOIS

1901





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May 30

1901

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mr. A. V. Miller B.S., 1897.

ENTITLED Use of the Compass in Geometrical  
Construction.

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Master of Science.

S. H. Shattuck

HEAD OF DEPARTMENT OF Mathematics



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P R E F A C E.

The Italian mathematician, Lorenzo Mascheroni, in his work, "La geometria del compasso," (Pavia 1797) has proven that all geometrical constructions can be made by means of the compass alone. This work was translated into the French by Carrette and from the French into the German by Grueson under the title "Gebrauch des Zirkles" (Berlin 1825). This problem has been very ably treated by Frischauf in his work, "Die geometrischen Constructionen von L. Mascheroni und J. Steiner", (Groz 1869) and also by Hutt in "Die Mascheroni'schen Konstruktionen" (Brandenburg 1880).

The corresponding problem, i.e., to make all geometrical constructions by means of the straight line alone, was first partly solved by the French mathematician, Brianchon in his "Application de la theorie des transversales" (Paris 1818). J. Steiner in his article "Die geometrischen Constructionen, ausgefuehrt mittelst der geraden Linie und eines festen Kreises" (Berlin 1833) has completely solved this problem with the help of a fixed circle.

These results are not generally known, and the object of the present work is to take the former problem, solve it and give a large number of examples as illustration of this method of construction. The work naturally divides itself into three sections as follows:

I. To determine what constructions are possible with the straight line and the compass together.

II. To prove that all these constructions can be made by means of the compass alone.

III. Examples. As examples for illustration, we take the construction problems in Wentworth's "Plane and Solid Geometry", revised edition.

With this as an outline we shall proceed with the development of it.

All references are to Wentworth's "Plane and Solid Geometry, Revised 1899".

Straight lines are determined by two points.





# CHAPTER I. GENERAL DISCUSSION OF THE PROBLEM.

## SECTION I.

To determine what constructions are possible with the straight line and compass.

1. In order to determine what constructions are possible with the straight edge and compass, we shall consider the possibilities of the case analytically. Take the two equations

$$a, x + b, y + c, = 0$$

$$ax^2 + ay^2 + bxy + cx + dy + e = 0,$$

which are the general equations of the straight line and the circle, and therefore, for particular values of the constants, represent analytically any single operation which can be made with the straight edge and the compass respectively. The mathematical magnitudes which can be constructed by operating with these equations would, at most, include only those involving operations of addition, subtraction, multiplication and division and the extraction of the square root. This latter enters only when the roots of the second equation are involved. When these roots are imaginary they are to be excluded, for imaginary quantities are not represented by ordinary geometrical constructions. We may conclude, therefore, that all the operations which we can perform by means of the straight edge and the compass are=

I. All rational operations.

II. The extracting of the square root of any number.

If now it can be shown that all the rational operations, as well as the extraction of the square root of any given quantity, can be performed by the compass alone, we have demonstrated our proposition.

## SECTION II.

To prove that all of the above mentioned constructions can be made with the compass alone.

2. In order to construct all of the rational operations mentioned in the last section, it will be necessary from time to time to make auxiliary constructions, such as bisecting a given arc, etc. These constructions will be introduced as they are needed.



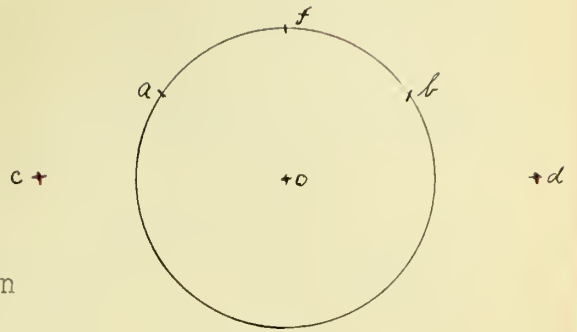


### 3. To bisect a given arc.

Let  $ab$  be the given arc of a circle whose centre is  $o$ .

To bisect  $ab$ .

With  $o$  and  $b$  as centres and  $ob$  and  $ab$  respectively as radii, describe arcs intersecting at  $d$ . Similarly with  $o$  and  $a$  as centres and  $oa$  and  $ab$  as radii locate the point  $c$ . Then  $aboc$  and  $abdo$  are parallelograms.



(W., Art., 182) Since  $od$  is parallel to  $ab$  is parallel to  $oc$ , then  $cd$  is a straight line, (W., Art., 105) and  $o$  is its middle point. Const.

With  $c$  and  $d$  as centres and radius  $cb$ , describe arcs intersecting at  $e$ .

Then  $eo$  is perpendicular to  $cd$  at its middle point. (W., Art., 161)

Hence  $oe$  is perpendicular to  $ab$ , (W., Art., 107)

and must bisect the arc  $ab$  at some point  $f$ . (W., Art., 245)

To determine the point  $f$ , we have from the parallelogram  $aboc$

$$\overline{cb}^2 + \overline{ao}^2 = 2\overline{ab}^2 + 2\overline{ob}^2, \quad (\text{W., Ex., 280})$$

or  $\overline{cb}^2 = 2\overline{ab}^2 + \overline{ob}^2.$

But  $\overline{cb}^2 = \overline{ce}^2 = 2\overline{ab}^2 + \overline{ob}^2,$

Const.

and  $\overline{co}^2 = \overline{ab}^2.$

Then  $\overline{ce}^2 - \overline{co}^2 = \overline{oe}^2 = \overline{ab}^2 + \overline{ob}^2,$

also  $\overline{cf}^2 = \overline{oc}^2 + \overline{of}^2 = \overline{ob}^2 + \overline{ab}^2.$

Hence  $\overline{oe}^2 = \overline{cf}^2$  or  $oe = cf.$

Therefore to determine the point  $f$ , make

$$cf = df = oe.$$

### 4. To find the intersection of a straight line and circle.

I. When the line passes through the centre of the circle.

Let  $ab$  be the given line and  $c$  the given circle.

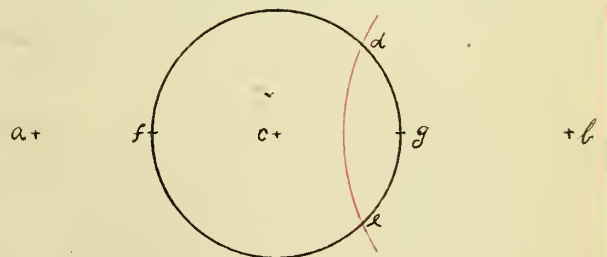
To find the intersection of  $ab$  with the circle  $c$ .

With  $b$  as a centre and any convenient radius strike an arc  $de$ . Then

$acb$  is perpendicular to  $de$  at its middle point, (W., Art., 264)

and therefore bisects the arcs  $de$  and

$dfe$ . (W., Art., 245)





By means of Article 3, bisect the arcs  $dge$  and  $dfe$  at the points  $g$  and  $f$ . Then  $g$  and  $f$  are the required points.

\*Cor. By means of this article we can lay off any distance on a given straight line.

II. When the line does not pass through the centre of the circle.

Let  $ab$  be the given line and  $c$  the given circle.

To find the intersections of  $ab$  with the circle  $c$ .

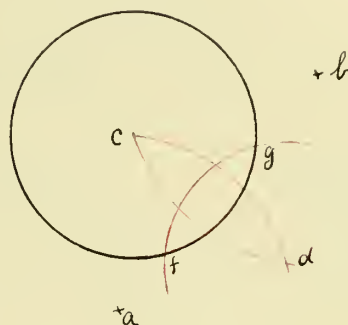
With  $a$  and  $b$  as centres and  $ac$  and  $bc$  respectively as radii describe arcs intersecting at  $d$ .

With  $d$  as a centre and the same radius as the given circle, describe an arc cutting the

given circumference at  $f$  and  $g$ . The two lines  $ab$  and  $fg$  are both perpendicular to  $cd$  at its middle point,

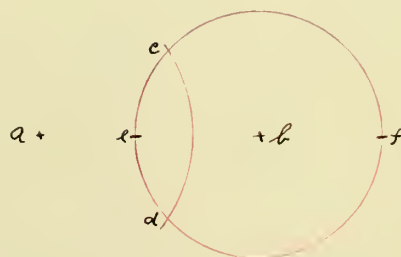
(W., Art., 161)

and must therefore be the same line. Therefore  $f$  and  $g$  are the required points.



5. Given two magnitudes represented by the straight lines  $A$  and  $B$ ; required to construct  $A+B$  and  $A-B$ .

+       $A$       +  
+       $B$       +



Let  $A$  and  $B$  be the given lines.

To construct  $A+B$  and  $A-B$ .

Take the line  $ab$  equal to  $A$  and with  $b$  as a centre and a radius equal to  $B$ , describe a circle.

Find the intersections  $e$  and  $f$  of this circle with the line  $ab$  (Art., 4). Then

$$ae = ab - be = A - B,$$

$$af = ab + bf = A + B.$$

Therefore  $ae$  and  $af$  are the required lines.

6. To construct a fourth proportional to three given straight lines.

Let  $m, n$  and  $p$  be the three given straight lines.

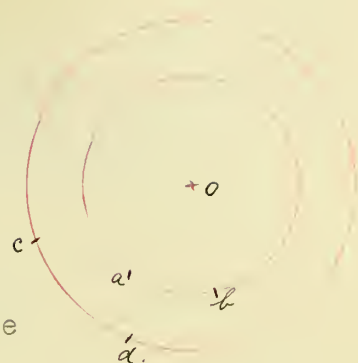
To construct a fourth proportional to  $m, n$  and  $p$ .





With  $m$  and  $n$  as radii and  $o$  as a centre, describe two concentric circles. Lay off  $cd=p$ . With  $c$  as a centre and any convenient radius cut the inner circumference at  $a$  and with  $d$  as a centre and the same radius cut it at  $b$ . Then  $ab$  is the required line.

+  $m$  +  
+  $n$  +  
+  $p$  +



Proof= In the triangles  $coa$  and  $dob$ , we have  
 $co=od$ ,  $oa=ob$  and  $ca=db$ .

Therefore, the triangles  $coa$  and  $dob$  are similar,

(W., Art., 358)

and hence  $\angle coa = \angle dob$ .

(W., Art., 351)

Adding  $\angle aod$  to both sides of this equation, we get  $\angle cod = \angle aob$ . Hence triangles  $aob$  and  $cod$  are similar,

(W., Art., 357)

and we have

$$oc:oa=cd:ab$$

or

$$m:n=p:ab.$$

Therefore  $ab$  is the required line.

\*Cor. If  $p$  is greater than  $2m$ , take equal multiples of  $m$  and  $n$  for radii.

7. Find the point of intersection of two straight lines.

Let  $ab$  and  $cd$  be the given straight lines.

To find the point of intersection of  $ab$  and  $cd$

With  $a$  as a centre and  $ad$  and  $ac$  as radii, and  $b$  as a centre and radii  $bc$  and  $bd$ , describe arcs intersecting at  $e$  and  $f$ . With  $d$  and

$f$  as centres and radii respectively equal to  $cf$  and  $dc$ , describe arcs intersecting at  $g$ . By means of the last article, construct a fourth proportional to  $eg$ ,  $ed$ , and  $fg$ . With  $e$  and  $d$  as centres and this fourth proportional as a radius, describe arcs intersecting at  $h$ . Then  $h$  is the required point.

Proof:  $ab$  is perpendicular to  $ed$  at its middle point, (W., Art., 161)  
and since  $eh=dh$ , then  $h$  lies on  $ab$ . (W., Art., 160)

Since  $eg:ed=fg:hd$

Const.

Then  $fg$  is parallel to  $hd$ .

(W., Art., 345)

But  $fg$  is parallel to  $cd$ ;

Const.

hence  $h$  must lie on  $cd$ . Therefore  $h$  is the intersection of  $ab$  and  $cd$





and is the required point.

8. Given two magnitudes represented by the straight lines A and B; find their product.

Let A and B be the given lines.

To construct  $A \times B$ .

Let  $ab=A$  and  $bc=B$  and construct

$ac=A+B$ . (Art., 5)

With b as a centre and unit radius describe a circle. Let d be any point in the circumference of this circle not in the line ac. Pass a circle through the three points a, d and c (W., Art., 258) Find the intersection, m, of the line bd with this circle. Then bm is the required product.

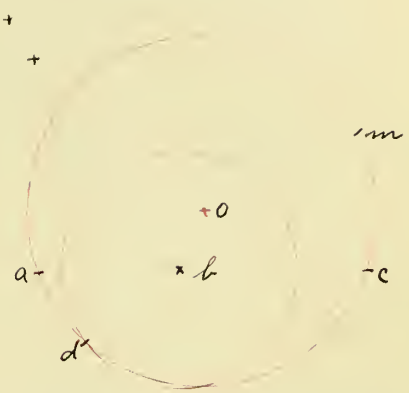
Proof:  $bm \times bd = ab \times bc$ .

But  $bd=1$

Then  $bm = ab \times bc = A \times B$ .

Therefore bm is the required line.

\*Cor. If  $A=B$ , then  $bm=A^2$ . By continuing the operation we can raise a number to any required power.



(W., Art., 378)

Const.

9. Given two magnitudes represented by the straight lines A and B; find their quotient.

Let A and B be the given lines.

To construct  $A/B$

Let  $ab=A$ . With b as a centre and a radius equal to unity, describe a circle  $M'$ , and let d be any point in the circumference of  $M'$  not in the line ab.

Pass a circle,  $M''$ , through the three points, a, d and c. (W., Art., 258)

Find the intersection, e, of bd with

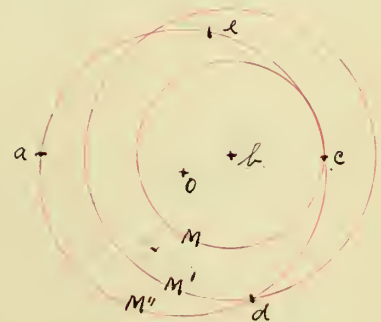
this circle. (Art., 4, II.) Then be is the required line.

Proof:  $ab \times bc = bd \times be$ .

(W., Art., 378)

But  $bc=1$ .

Then  $ab = bd \times be$  or  $A = B \times be$  and  $be = A/B$ .







Therefore  $be$  is the required line.

10. Given a magnitude represented by the straight line  $A$ ; construct the square root of  $A$ .

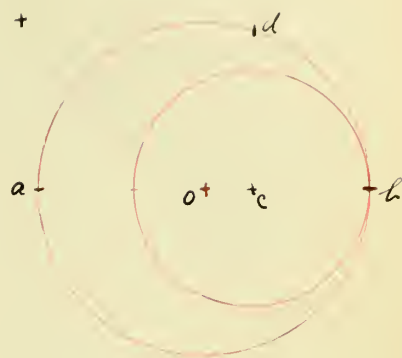
Let  $A$  be the given line.

+      $A$      +

To construct the square root of  $A$ .

Let the given line,  $A$ , be laid off in the position  $ac$ . With  $c$  as a centre and a unit radius, construct a circle. Find the intersection,  $b$ , of  $ac$  with this circle.

(Art. 4, I) Construct a circle on  $ab$  as a diameter. (W., Art., 302) Erect a perpendicular to the line  $ab$  at the point  $c$ , intersecting this circumference at  $d$ . (W., Art., 301) Then  $cd$  is the required line.



Proof:  $ac:cd=cd:cb$ .

(W., Art., 370)

But  $cb=1$ .

Const.

Then  $cd^2=ac=A$  or  $cd=\sqrt{A}$ .

Therefore  $cd$  is the required line.

11. We have now constructed by means of the compass alone all the rational operations (addition, subtraction, multiplication and division) and the extraction of the square root. In section I we found that these are all operations which can be constructed by means of both the compass and the straight edge. We have therefore demonstrated our proposition that all the constructions which can be made by means of both the compass and the straight edge can be made by means of the compass alone.



## CHAPTER II. APPLICATIONS.

The examples of the present chapter are the construction problems of Wentworth's "Plane and Solid Geometry", Revised 1899.

12. To erect a perpendicular to a line at a given point.

I. When the point is on the line.

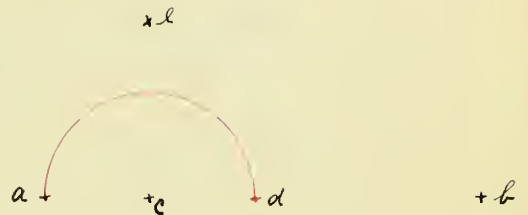
Let  $ab$  be the given line and  $c$  the given point.

To erect a perpendicular to  $ab$  at  $c$ .

With  $c$  as a centre and  $ca$  as a radius, describe a circle and find its intersection,  $d$ , with  $ab$ . (Art., 4, I)

With  $a$  and  $d$  as centres and any convenient radius, describe arcs intersecting at  $e$ . Then  $ec$  is the required perpendicular.

Proof:  $ac = cd$ , (W., Art., 217)  
and  $ae = de$ . Const.  
Therefore  $ec$  is perpendicular to  $ab$ . (W., Art., 161)

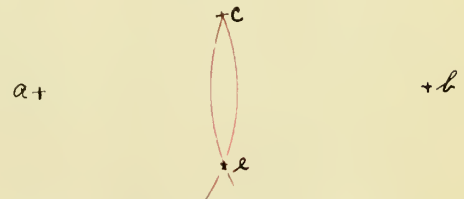


II. When the point is without the line.

Let  $ab$  be the given line and  $c$  the given point.

To erect a perpendicular to  $ab$  from  $c$ .

With  $a$  and  $b$  as centres and  $ac$  and  $bc$  respectively as radii, describe arcs intersecting at  $e$ . Then  $ec$  is the required perpendicular. (W., Art., 161)



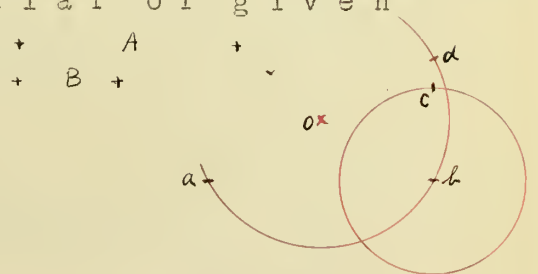
13. Given a line  $A$ , to construct at its end point a perpendicular of given length  $B$ .

Let  $A$  and  $B$  be the given lines.

To construct a perpendicular of length  $B$  at the end point of  $A$ .

Take the line  $ab$  equal to  $A$ .

With  $a$  and  $b$  as centres and any convenient radius, describe arcs intersecting at  $o$ . With  $o$  as a cent-







re and the same radius, describe a circle. Find the intersection, d, of this circle with ao (Art., 4I). With b as a centre and B as a radius, describe a circle and find its intersection, c, with bd. (Art., 4, I). Then bc is the required perpendicular.

Proof: bd is perpendicular to ab

(W., Art., 290)

and

bc=B.

Const.

Therefore bc is the required perpendicular.

\*Cor. 1. From the right triangle abc, we get  $\overline{ac}^2 = \overline{ab}^2 + \overline{bc}^2$ , i.e.  $\overline{ac}^2 = A^2 + B^2$ .

By extracting the square root of both sides of the equation, we have

$ac = \sqrt{A^2 + B^2}$ .

By taking A as the diameter of a circle and from one extremity of it laying off B as a chord, we would get  $ac = \sqrt{A^2 - B^2}$ .

#### 14. To bisect a given line. (Hutt)

+ A +

Let A be the given line.

To bisect A.

With a as a centre and A as a radius, describe a circle and let b be any point in the circumference. With c as a centre and co as a radius, strike an arc cutting the first circumference at d and e. With e and d as centres and radius A, describe arcs intersecting at o. Then ab, and therefore A, is bisected at o.

Proof: Triangles ado and adc are similar.

(W., Art., 357)

Hence

$ao:ad=ad:ac$ ,

and therefore  $ao=\overline{ad}^2/\overline{ac}=\overline{ab}^2/2ab=ab/2=A/2$ .

Therefore A is bisected at o.

\*Cor. To bisect ob.

Make  $ag=af=bd$  and  $go'=fo'=bd$ .

Then ob is bisected at o'.

Proof: Triangles ago' and age are similar.

(W., Art., 357)

Then

$ao':ag=ag:ac$  therefore  $ao'=\overline{ag}^2/\overline{ac}=\overline{bd}^2/\overline{ac}$ .

Since ac is bisected at b, we have

$\overline{ad}^2 + \overline{cd}^2 = 2\overline{ab}^2 + 2\overline{bd}^2$ .

(W., Art., 377)

But

$\overline{ad}^2 + \overline{cd}^2 = 5\overline{ab}^2$ .

Hence

$5\overline{ab}^2 = 2\overline{ab}^2 + 2\overline{bd}^2$  or  $\overline{bd}^2 = 3/2 \overline{ab}^2$ .

substituting we have

$ao' = (3/2 \overline{ab}^2) / 2ab = 3/4 ab$ .

Therefore

$bo' = 1/4 ab$ .

Proceeding in a similar manner we can bisect bo'.



# 15. To bisect a given angle.

Let  $abc$  be the given angle.

To bisect the angle  $abc$ .

With  $b$  as a centre and  $ba$  as a radius, describe an arc and find where it intersects  $bc$ . (Art., 4, I). With  $a$  and  $d$  as centres and any convenient radius describe arcs intersecting at  $e$ . Then  $be$  bisects the given angle  $abc$ . (W., Art., 245)



# 16. To construct an angle of $45^\circ$ and $135^\circ$ .

Take any straight line,  $ab$ , and erect a perpendicular,  $cd$ , at its middle point. (Art., 12, I)

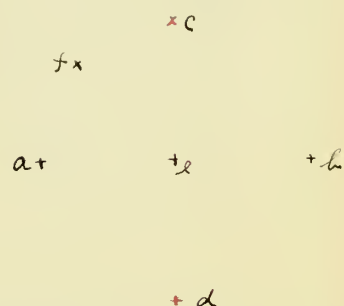
Find the intersection,  $e$ , of  $ab$  and  $cd$ . (Art., 7)

Bisect the angle  $aec$  by the line  $ef$ . (Art., 15)

Then  $\angle aef = \frac{1}{2} \angle aec = \frac{1}{2} \text{ of } 90^\circ = 45^\circ$ ,

and  $\angle bef = 180^\circ - \angle aef = 135^\circ$

Therefore  $aef$  and  $bef$  are the required angles.



# 17. Construct an equilateral triangle, having given one side.

Let  $a$  be the given side.

To construct an equilateral triangle, having its sides equal to  $a$ .

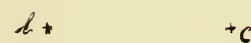
Take any line,  $bc$ , equal to  $a$ .

With  $b$  and  $c$  as centres and  $a$  as a radius, strike arcs intersecting at  $d$ . Then  $bcd$  is the required triangle for each side is equal to  $a$ .

\*Cor.  $\angle dbc = \frac{1}{3} \text{ of } 180^\circ = 60^\circ$

Draw  $be$  perpendicular to  $bd$  at  $b$ .

Then  $\angle cbe = 60^\circ + 90^\circ = 150^\circ$



# 18. To trisect a right angle.

Let  $acd$  be the given right angle.

To trisect the angle  $acd$

With  $a$  and  $d$  as centres and  $ac$  as a radius cut the circumference at  $e$  and  $f$ . Then the angle  $acd$  is trisected by  $e$  and  $f$ .

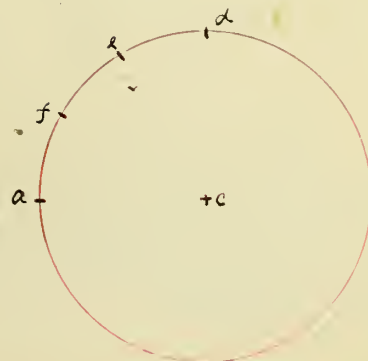
Proof:  $\angle acd = 90^\circ$  Hypothesis.

and  $\angle ace = 60^\circ$ . (Cor., Art., 17)

Therefore  $\angle ecd = 90^\circ - 60^\circ = 30^\circ$ .

Similarly  $\angle acf = 30^\circ$ ,

and therefore  $\angle ecf = 30^\circ$ .







Hence  $\angle acd$  is trisected.

19. At a given point in a given straight line to construct an angle equal to a given angle.

Let  $abc$  be the given angle,  $b'c'$  the given line and  $b'$  the point on the line.

To construct at  $b'$  an angle equal to  $abc$ .

With  $b$  as a centre and  $ba$  as a radius, strike an arc and find its intersection,  $e$ , with  $bc$ . (Art., 4, I). With  $b'$  as a centre and the same radius strike an arc cutting  $b'c'$  at  $e'$ . With  $e'$  as a centre and  $ae$  as a radius strike an arc locating the point  $a'$ . Then  $a'b'c'$  is the required angle.

Proof: Chord  $ae = \text{chord } a'e'$ .

Const.

Then  $\angle a'b'c' = \angle abc$ .

(W., Arts., 243, 237)

Therefore  $a'b'c'$  is the required angle.



20. To draw a straight line parallel to a given straight line through a given external point.

Let  $ab$  be the given line and  $c$  the given point.

To draw a line through  $c$  parallel to  $ab$ .

With  $b$  and  $c$  as centres and  $ac$  and  $ab$  respectively as radii, describe arcs

intersecting at  $d$ . Then  $cd$  is the required line.

Proof:  $ab = cd$  and  $ac = bd$ ,

Const.

therefore  $abdc$  is a parallelogram.

(W., Art., 182)

Hence  $cd$  is parallel to  $ab$  and is the required line.

$c +$   $+ d$

$a +$   $+ b$

21. To multiply a given line by any positive integer.

Let  $ab$  be the given line.

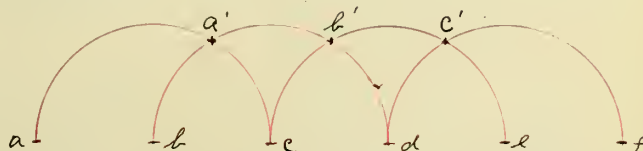
To multiply  $ab$  by any positive integer.

With  $b$  as a centre and  $ba$  as a radius, draw a circle and find its intersection,  $c$ , with  $ab$ . (Art., 4, I). Now with  $c$  as a centre and the same radius, draw a circle cutting  $ac$  at  $d$ . This can be continued indefinitely.

Then  $ac = 2ab$ .

(W., Art., 217)

Similarly  $ad = 3ab$  etc.





\*Cor. The points a'b'c' etc., form a similar series of points, with  $a'b'=ab$ .

22. To divide a given straight line into a given number of equal parts.

Let  $ab$  be the given straight line.

To divide  $ab$  into equal parts.

Construct  $ac=n \times ab$ . (Art., 21)

With  $a$  and  $c$  as centres and radii  $ab$  and  $ac$ , describe arcs intersecting in  $d$  and  $f$ . With  $f$  and  $c$  as centres and radii  $ab$  and  $fd$  respectively, describe arcs intersecting at  $e$ . Then  $ae=ab/n$ .

Proof: Both  $ac$  and  $ec$  are parallel to  $fd$ .

(W., Art., 182)

Then  $e$  must lie on the line  $ac$ .

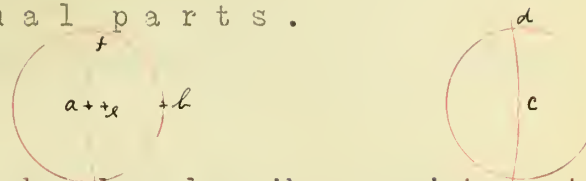
The triangles  $afe$  and  $afc$  are similar

(W., Art., 357)

Therefore  $ae:af=af:ac$

or  $ae:ab=ab:ac=1:n$

Hence  $ae=ab/n$



23. To construct an equilateral triangle having given the perimeter.

Let  $ab$  be the given perimeter.

To construct an equilateral triangle, having given the perimeter,  $ab$ .

Divide  $ab$  at  $c$  and  $d$  into three equal parts. (Art., 22).

Construct an equilateral triangle, having  $cd$  as a side. (Art., 17)

Then  $cde$  is the required triangle.

Proof: Triangle  $cde$  is equilateral.

Its perimeter is  $3 \times cd = ab$ .

Therefore  $cde$  is the required triangle.

Second method.

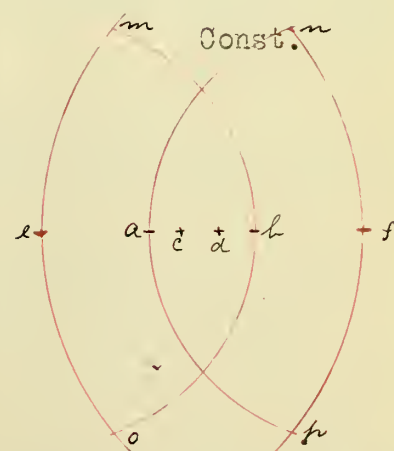
Let  $ab$  be the given perimeter.

To construct an equilateral triangle, having  $ab$  as a perimeter.

Make  $ae=bf=ab$  (Art., 21)

With  $e$  and  $f$  as centres and radii  $eb$  and  $ef$ , describe arcs intersecting at  $m, n, o$

and  $p$ . With  $o$  and  $m$  as centres and radius  $eb$ , describe arcs intersecting in  $c$ , and with the same radius and  $n$  and  $p$  as centres, describe arcs intersecting in  $d$ . Then  $ab$  is trisected at  $c$  and  $d$ , and an equilateral triangle with  $ac$  as a side will be the required triangle.







Proof: The points a,c,d,b lie in the same straight line (W.,Art.,160)  
The triangles emc and emf are similar. (W.,Art.,357)

Then  $em:ec=ef:em=3:2$

Hence  $em=2ab=3/2(ab+ac)$  or  $ab=3ac$

Similarly  $ab=3bd$  or  $ab=3cd$

Therefore the triangle constructed with ac as a side is the required triangle.

24. To divide a line into four equal parts by two different methods.

A line may be divided into four equal parts by applying article 14 twice. In that article, if we had bisected ao as we did bo, then ab would have been divided into four equal parts.

This problem may be solved by using article 22. In this case  $n=4$  and  $ac=4ab$

25. Through a given point to draw a line which shall make equal angles with the two sides of a given angle.

Let abc be the given angle  
and p the given point.

To draw a line through p  
making equal angles with ab and cb.

Draw the bisector, bd, of the  
angle abc. Construct pp' perpendicular to bd. (Arts., 15 & 12, II)  
Then pp' is the required line.

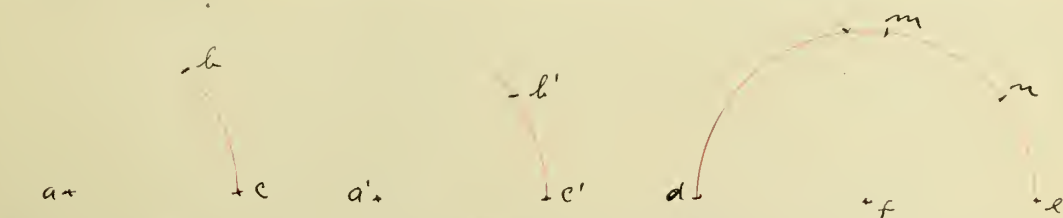
Proof: The right triangles bgf and bef are equal. (W.,Art.,142)

Hence  $\angle bgf = \angle bef$ . (W.,Art.,128)

Therefore pp' is the required line.

\*Cor. This also solves the problem: To draw a line through a given point so that it shall form with the sides of a given angle an isosceles triangle.

26. To find the third angle of a triangle when two of the angles are given.



Let bac and b'a'c' be the two given angles.

To construct the third angle of the triangle.



At any point,  $f$ , in the line  $de$  construct

$$\angle efn = \angle b'a'c' \text{ and } \angle nfm = \angle bac. (\text{Art.}, 19)$$

Then  $\angle dfm$  is the required angle. (W., Art., 139)

27. To construct a triangle when two sides and the included angle are given.



Let  $a$  and  $b$  be the given sides and  $b'a'c'$  the given angle.

To construct a triangle having the given sides and angles.

On the line  $cd$ , construct  $\angle fcd = \angle b'a'c'$ . (Art., 19)

With  $c$  as a centre and radius  $a$ , strike an arc and find its intersection  $e$ , with  $cf$ . (Art., 4, I)

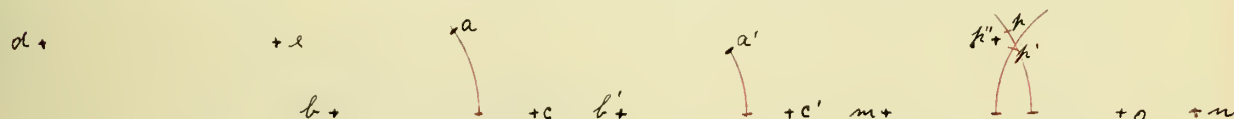
Similarly lay off  $cg$  equal to  $b$ .

Then  $ceg$  is the required triangle.

Proof:  $\angle fcd = \angle b'a'c'$ ,  $ce = a$  and  $cg = b$ . Const.

Therefore  $ceg$  is the required triangle. (W., Art., 143)

28. To construct a triangle when a side and two angles of the triangle are given.



Let  $de$  be the given side and  $abc$  and  $a'b'c'$  the given angles.

To construct a triangle having  $de$  for one of its sides and  $abc$  and  $a'b'c'$  as two of its angles.

On any line  $mn$  lay off  $mo = de$ . (Art., 4, I)

At  $m$  and  $o$  respectively construct  $\angle pmo = \angle abc$  and  $\angle p'om = \angle a'b'c'$  (Art., 19)

Find the intersection,  $p$ , of  $mp$  and  $op'$ . (Art., 7)

Then  $mop$  is the required triangle.

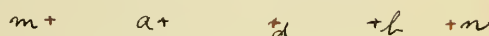
Proof:  $mo = de$ ,  $\angle pmo = \angle abc$  and  $\angle p'om = \angle a'b'c'$  Const.

Therefore  $mop$  is the required triangle. (W., Art., 139)

29. To construct an equilateral triangle having given the altitude.

Let  $p$  be the given altitude.

To construct an equilateral triangle having  $p$  for its altitude.





At any point d of the line mn, draw dc perpendicular to mn and lay it off equal to p. (Arts., 12, 4, I)

At c, construct  $\angle acd = \angle bcd = 30^\circ$  (Art., 18)

and find the intersections, a and b, with mn. (Art., 7)

Then abc is the required triangle.

Proof: The altitude  $dc = p$ . Const.

Tri.  $adc = \text{tri. } bdc$ . (W., Art., 142)

Hence  $\angle dac = \angle dbc$ . (W., Art., 127)

But  $\angle acb = 60^\circ$  Const.

and therefore  $\angle dac + \angle dbc = 2\angle dac$  or  $2\angle dbc = 180^\circ - 60^\circ = 120^\circ$ ,

or  $\angle dac = \angle dbc = 60^\circ$ .

then triangle abc is equiangular and therefore equilateral (W., Art., 148)

Therefore abc is the required angle.

30. To construct an isosceles triangle having given:

I. The base and altitude.

Let b be the given base and a the given altitude.

To construct an isosceles triangle with base b and altitude a.

On any line ch, lay off  $cd = b$  (Art., 4, I) \* f

and erect a perpendicular fg at its middle point, (Art., 12, I) \* e

Lay off  $ge = a$ . (Art., 4, I)

Then cde is the required triangle. c + g + d + h

Proof: The altitude  $ge = a$  Const.

and  $ce = de$ . (W., Art., 160)

Therefore cde is the required triangle.

II. The altitude and one of the legs.

Let b' be the given leg and + d

a' the given altitude. + e

+ a' +

To construct an isosceles triangle having a' as an altitude and b' as one of the equal sides.

m + a + d + b + n

At any point in the line mn, erect a perpendicular de. (Art., 12, I)

Lay off  $de = a'$  (Art., 4, II)

With c as a centre and a radius b', strike an arc and find its points of intersection a and b, with mn. (Art., 4, II)

Then abc is the required triangle.

Proof: The altitude  $cd = a'$  and  $ac = bc = b'$  Const.

Therefore abc is the required triangle.





III. The angle at the vertex and the altitude.

Let  $a$  be the given altitude and  $\angle bcd$  the given angle at the vertex.

To construct an isosceles triangle whose altitude is  $a$  and whose vertical angle is  $\angle bcd$ .

Erect  $op$  perpendicular to  $mn$  at any point and lay off  $op' = a$ .

Bisect the angle  $\angle bcd$ ,

and at  $p'$  construct half the angle  $\angle bcd$  on each side of  $op'$ .

Find the points of intersection,  $m'$  and  $n'$ , of the sides of these angles with the line  $mn$ .

Then  $m'n'p'$  is the required triangle.

Proof: The altitude  $op' = a$

and  $\angle m'p'n' = \angle bcd$ .

Also  $\text{Tri. } m'op' = \text{tri. } n'op'$

and therefore  $m'p' = n'p'$ .

Hence the triangle  $m'n'p'$  is isosceles.

Therefore  $m'n'p'$  is the required triangle.

(Arts., 12, I; 4, I)

(Art., 15)

Const.

Const.

(W., Art., 142)

(W., Art., 127)

(W., Art., 120)

31. To construct a triangle when two sides and the angle opposite one of them are given.

Let  $a$  and  $b$  be the two given sides and  $\angle cde$  the given angle opposite the side  $a$ .

To construct a triangle having  $a$  and  $b$  as two of its sides and the angle opposite  $a$  equal to  $\angle cde$ .

At the point  $m$  in the line  $mn$ , construct angle  $\angle pmn$  equal to angle  $\angle cde$ .

Lay off  $ma'$  equal to the given side  $b$ .

With  $a$  as a centre and  $a$  as a radius, strike an arc and find its intersection,  $b'$  and  $b''$ , with  $mn$ .

Then either  $ma'b'$  or  $ma'b''$  is the required triangle.

Proof:  $a = a'b' = a'b''$ ;  $b = ma'$ ;  $\angle cde = \angle a'mb'$ .

Therefore  $ma'b'$  or  $ma'b''$  is the required triangle..

\*Cor. If  $a$  is of such a length that the arc is tangent to  $mn$  or if  $a = b$  we have but one triangle. If  $a$  is less than the distance from  $a'$  to  $mn$ , we have no construction.



32. To construct a triangle when the three sides of the triangle are given.

The triangle may be constructed by laying any one of the given sides on any line, taking these two points as centres and the other two given sides as radii and strike arcs locating the third vertex of the required triangle.

33 To construct a parallelogram when two sides and the included angle are given.

Let  $a$  and  $b$  be the given sides and  $\angle b'c'$  the given angle.

To construct a parallelogram having its two adjacent sides equal respectively to  $a$  and  $b$  and the included angle equal to  $\angle a'b'c'$ .

On any line,  $mn$ , lay off  $mn' = b$  and construct  $\angle emn' = \angle a'b'c'$ . (Arts., 4, Cor; 19) Lay off  $me = a$ . With  $e$  and  $n'$  as centres and radii equal respectively to  $b$  and  $a$ , describe arcs intersecting in  $d$ . Then  $mn'de$  is the required parallelogram.

Proof:  $mn' = b$ ,  $me = a$  and  $\angle emn' = \angle a'b'c'$ ,  
also  $ed = mn'$  and  $n'd = me$ ,  
hence  $mn'de$  is the required parallelogram.

Const.

Const.

34. To circumscribe a circle about a given triangle.

Let  $abc$  be the given triangle.

To circumscribe a circle about  $abc$ .

Erect perpendiculars  $ed$  and  $fg$  at the middle points of  $bc$  and  $ac$ . (Art., 12, I)

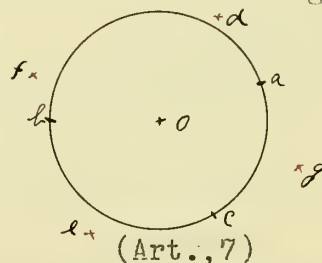
Find the intersection  $o$  of  $ed$  and  $fg$ .

With  $o$  as a centre and  $oa$  as a radius, describe a circle. Then this is the required circle.

Proof: Since the point  $o$  lies on both lines  $ed$  and  $fg$  it is equally distant from  $a, b$  and  $c$ .

(W., Art., 160)

Then the circle with  $o$  as centre and radius  $oa$  will pass through  $a, b$  and  $c$ . Therefore this is the required circle.

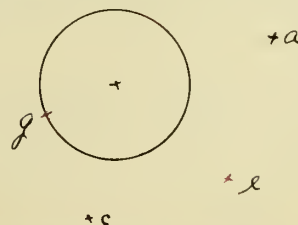


35. To inscribe a circle in a given triangle.

Let  $abc$  be the given triangle.

To inscribe a circle in  $abc$ .

Bisect the angles  $b$  and  $c$  by the







lines be and cd.

(Art., 19)

Find the intersection o of cd and be.

(Art., 7)

From o draw og perpendicular to bc.

(Art., 12, II)

Find the intersection, g, of og and bc.

(Art., 7)

With o as a centre and og as a radius, describe a circle.

Then this is the required circle.

Proof: Since o is on both be and cd it is equally distant from all three sides of the triangle abc.

(W., Art., 162)

Hence a circle described with o as a centre and og as a radius will touch all three sides of the triangle. Therefore this is the required circle.

36. Through a given point to draw a tangent to a given circle.

I. When the given point is on the circumference.

Let c be the point on the circumference whose centre is o.

To draw a tangent to the circumference at c.

Erect a perpendicular ac to oc at the point c.

(Art., 12, I)

Then ac is the required tangent.

(W., Art., 253)

II. When the given point is without the circle.

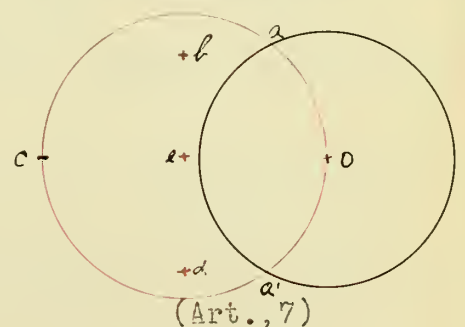
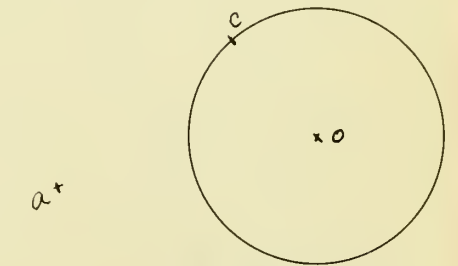
Let o be the centre of the given circle and c the given point.

To draw a tangent to the circle whose centre is o from the given point c.

Erect a perpendicular, bd, at the middle point of oc.

(Art., 12, I)

Find the intersection, e, of oc and bd.



With e as a centre and ec as a radius, describe a circle cutting the given circumference at a and a'.

Then ac and a'c are the required tangents.

Proof: ca is perpendicular to oa and ca' to oa' (W., Art., 290)

Therefore ca and ca' are tangents to the given circle. (W., Art., 253)

37. To draw a tangent to a given circle, so that it shall be parallel to a given straight line.



Let  $o$  be the centre of the given circle,  $+g$   
and  $ab$  the given straight line.

To draw a tangent to the circle whose  
centre is  $o$  which is parallel to  $ab$ .

From  $c$ , draw  $oc$  perpendicular to  $ab$ .

Find the intersections,  $e$  and  $d$ , of  $oc$  with the  
circumference. (Art., 12, II) & (Art., 4, I)  $a+$

Draw tangents to the circle at the points  $e$  and  $d$ .

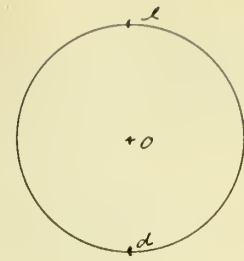
Then these are the required tangents.

Proof:  $oc$  is perpendicular to  $ab$ .

Also  $ge$  and  $df$  are perpendicular to  $ce$ .

Then  $ge, df$  and  $ab$  are all parallel.

Therefore  $ge$  and  $df$  are the required tangents.



(Art., 36, I)  $+f$   
 $+b$

Const.

(W., Art., 254)

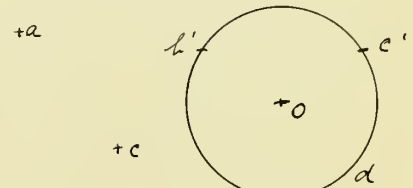
(W., Art., 104)

38. Upon a given straight line, to de-  
scribe a segment of a circle in which a  
given angle can be inscribed.  $+a'$

Let  $abc$  be the given angle. and

$b'c'$  the given line.

To describe the segment of a  
circle on  $b'c'$  in which angle  $abc$  may  
be inscribed.  $+a$   $+c$



(Art., 19)

At  $b'$  construct  $\angle a'b'c' = \angle abc$

Erect a perpendicular to  $a'b'$  and to  $b'c'$  at its middle point, (Art., 12, I)  
and find the intersection,  $o$ , of these perpendiculars. (Art., 7)

With  $o$  as a centre and  $ob'$  as a radius, describe a circle. Then  $b'dc'$  is  
the required segment.

Proof: Any angle inscribed in the segment  $b'dc'$  is measured by one-  
half the arc  $b'c'$ . (W., Art., 289)

Also  $\angle a'b'c' = \angle abc$  is measured by half the arc  $b'c'$ . (W., Art., 295)

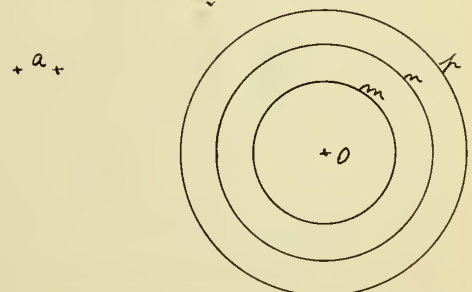
Then any angle inscribed in the segment  $b'dc'$  is equal to the angle  $abc$ .  
Therefore  $b'dc'$  is the required segment..

39. Find the locus of a point at a given  
distance from a given circumference.

Let  $a$  be the given distance and  
 $n$  the given circumference with centre  
 $o$ .

To find the locus of points at a  
distance  $a$  from the circumference  $n$ .

Let  $r$  be the radius of the cir-





cumference  $n$  and construct  $r+a$  and  $r-a$ .

(Art., 5)

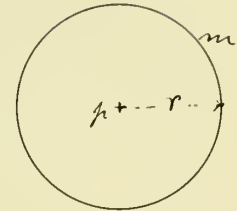
With  $o$  as a centre and  $r-a$  and  $r+a$  as radii, describe the circumference  $m$  and  $p$ . Then  $m$  and  $p$  are the required loci.

40. Find the locus of the centre of a circle,

I. Which has a given radius  $r$  and passes through a given point  $p$ .

Let  $p$  be the given point and  $r$  the given radius.

To find the locus of the centre of the circle whose radius is  $r$  and whose circumference passes through  $p$ .



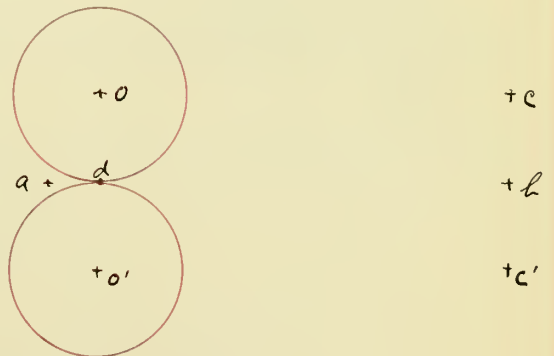
With  $p$  as a centre and  $r$  as a radius, describe a circumference  $m$ . Then this is the required locus.

Proof: If we take any point in  $m$  as a centre and  $r$  for a radius and describe a circle, the circumference of this circle will pass through  $p$ . Therefore  $m$  is the required locus.

II. Which has a given radius and touches a given line  $ab$ .

Let  $ab$  be the given line and  $r$  the given radius.

To find the locus of the centre of a circle whose radius is  $r$  and whose circumference touches the line  $ab$ .



At  $d$ , draw  $oo'$  perpendicular to  $ab$ .

(Art., 12, I)

Lay off  $od = o'd = r$ .

(Art., 4, Cor.)

Through  $o$  and  $c'$  draw  $oc$  and  $o'c'$  respectively parallel to  $ab$ . (Art., 20)

Then  $oc$  and  $o'c'$  are the required loci.

Proof: Every point in  $oc$  and  $o'c'$  is at a distance  $r$  from  $ab$ .

(W., Art., 181)

Then if we take any point in  $oc$  or  $o'c'$  as a centre and  $r$  as a radius and describe a circle, then the circumference of this circle will touch  $ab$ . Therefore  $oc$  and  $o'c'$  are the required loci.

III. Which passes through two given points  $p$  and  $q$ .

Let  $p$  and  $q$  be the two given points.

To find the locus of the centres of all circles whose circumferences pass through  $p$  and  $q$ .

$h+$

$+g$

$+h$





Erect a perpendicular,  $ab$ , at the middle point of  $pq$ . (Art., 12, I)  
Then  $ab$  is the required locus.

Proof: Any point in  $ab$  is equally distant from the two points  $p$  and  $q$ .  
(W., Art., 160)

Then any circumference which passes through the two points  $p$  and  $q$  will have its centre in the line  $ab$ .

Therefore  $ab$  is the required locus.

IV. Which touches a given straight line  $ab$  at a given point  $p$ .

Let  $ab$  be the given line and  $p$  the given point.

To find the locus of the centre of a circle which is tangent to  $ab$  at the point  $p$ .

Construct  $cd$  perpendicular to  $ab$  at  $p$ . (Art., 12, I)

Then  $cd$  is the required locus.

Proof: All circles whose centres are in  $cd$  and whose circumferences pass through  $p$  are tangent to  $ab$  at  $p$ . (W., Art., 253)

A circle whose centre is not on  $cd$  and whose circumference passes through  $p$  is not tangent to  $ab$  at  $p$ . (W., Art., 254)

Therefore  $cd$  is the required locus.

V. Which touches each of two given parallel lines.

Let  $ab$  and  $cd$  be the two given parallel lines.

To find the locus of the centre of a circle which touches  $ab$  and  $cd$ .

Construct  $ef$  midway between and parallel to  $ab$  and  $cd$ . (Art., 20)

Then  $ef$  is the required locus.

Proof: Since every point of  $ef$  is equally distant from  $ab$  and  $cd$ , a circle described with any point  $p$  of  $ef$  as a centre and the distance from  $p$  to  $ab$  or  $cd$  as a radius will touch both  $ab$  and  $cd$ . This will not be true for any centre outside of  $ef$ .

Therefore  $ef$  is the required locus.

VI. Which touches each of two intersecting lines.

Let  $ab$  and  $cd$  be the given intersecting lines.

To find the locus of the centre of a circle tangent to  $ab$  and  $cd$ .





Find the intersection,  $o$ , of  $ab$  and  $cd$ . (Art., 7)  $a, +f, +d$   
 Construct the bisectors  $ef$  and  $e'f'$  of the  
 angles  $aoc$  and  $aod$ . (Art., 15)  $e, +, +o, +f$   
 Then  $ef$  and  $e'f'$  are the required loci.

Proof: The lines  $ef$  and  $e'f'$  are the loci of  $c, +, +h$   
 points equally distant from  $ab$  and  $cd$ . (W., Art., 162)  $+e'$   
 Then any circle tangent to  $ab$  and  $cd$  will have its centre in  $ef$  or  $e'f'$ .  
 Therefore  $ef$  and  $e'f'$  are the required loci.

41. To find in a given line a point  $x$  which  
 is equidistant from two given points.

Let  $ab$  be the given line and  $c$  and  $d$  the  $+f$   
 given points.  $a, +, x, +h$

To find a point on  $ab$  equidistant from  $c$  and  $d$ .  $c, +$

Construct  $ef$  perpendicular to  $cd$  at its mid-  $+d$   
 dle point. (Art., 12, I)  $+e$

Find the intersection  $x$  of  $ab$  and  $ef$ . Then  $x$  is the required point. (Art., 7)

Proof: Since  $x$  lies in  $ef$  it is equidistant from  $c$  and  $d$ . (W., Art., 160)  
 $x$  also lies in  $ab$  and is therefore the required point.

42. To find a point  $x$  equidistant from  
 three given points.

This problem is solved in article 35.

43. To find a point  $x$  equidistant from two  
 given points and at a given distance from  
 a third given point.

Let  $a$  and  $b$  be the first two given  $a, +, r, +$   
 points,  $c$  the third and  $r$  the given dis-  
 tance.

To find a point equidistant from  $a$  and  
 $b$  and at a distance  $r$  from  $c$ .  $e, +, h, +$

With  $c$  as a centre and  $r$  as a radius describe a circle. Construct  $ef$  per-  
 pendicular to  $ab$  at its middle point. (Art., 12, I)

Find its intersections,  $x$  and  $x'$ , with the circle whose centre is  $c$ . (Art., 4, II)  
 Then  $x$  and  $x'$  are the required points.

Proof: Since the points  $x$  and  $x'$  are on the circumference of the circle,  
 they are at the distance  $r$  from  $c$ . Since they lie in the line  $ef$  they  
 are equidistant from  $a$  and  $b$ . (W., Art., 160)



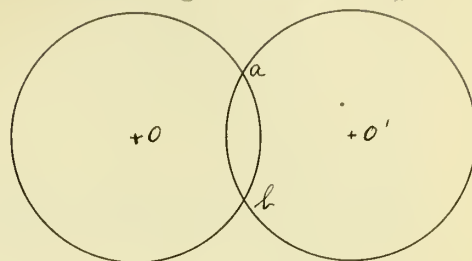


Therefore  $x$  and  $x'$  are the required points.

44. To construct a circle which has a given radius and passes through two given points.

Let  $r$  be the given radius and  $a$  and  $b$  the given points.

To construct a circle with radius  $r$  passing through the given points  $a$  and  $b$ .



With  $a$  and  $b$  as centres and radius  $r$ , describe arcs intersecting at  $o$  and  $o'$ . With  $o$  and  $o'$  as centres and  $r$  as a radius, describe circles. Then these are the required circles.

Proof: Since  $bo=ao=r$ , then a circle described with  $o$  as a centre and  $r$  as a radius will pass through  $a$  and  $b$ . Similarly for  $o'$ . Therefore the circles whose centres  $o$  and  $o'$  are the required circles.

45. To find a point  $x$  at a given distance from two given points.

The points  $o$  and  $o'$  of article 44 were so constructed.

46. To construct a circle which has its centre in a given line and passes through two given points.

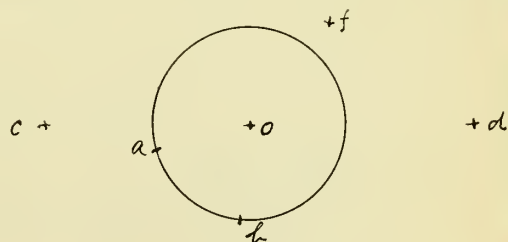
Let  $cd$  be the given line and  $a$  and  $b$  the given points.

To construct a circle passing through  $a$  and  $b$  whose centre lies in  $cd$ .

Construct  $ef$  perpendicular to  $ab$  at its middle point.

Find the intersection  $o$  of  $ef$  and  $cd$

With  $o$  as a centre and  $oa$  as a radius, describe a circle. Then this is the required circle.



(Art., 12, I)

(Art., 7)

Proof: Since  $o$  lies in  $ef$  it is equidistant from  $a$  and  $b$ .

(W., Art., 160)

Then if we take  $o$  as a centre and  $oa$  as a radius the circle will pass through  $a$  and  $b$ . Therefore this is the required circle.

47. To find a point  $x$  equidistant from two given points and also equidistant from two intersecting lines.





Let  $p$  and  $p'$  be the given points  
and  $ab$  and  $cd$  the given intersecting  
lines.

To find a point  $x$  equidistant  
from the lines  $ab$  and  $cd$  and the points  
 $p$  and  $p'$ .

Find the intersection,  $o$ , of  $ab$  and  $cd$ .

Construct the bisector,  $ef$ , of the angle  $bod$ .

Construct  $e'f'$  perpendicular to  $pp'$  at its middle point.

Find the intersection,  $x$ , of  $ef$  and  $e'f'$ .

Then  $x$  is the required point.

Proof: Since  $x$  lies on  $ef$  it is equidistant from  $ab$  and  $cd$ .

(W., Art., 162)

And since it lies on  $e'f'$  it is equidistant from  $p$  and  $p'$ . (W., Art., 160)

Therefore  $x$  is the required point.

48 To find a point  $x$  equidistant from  
two given points and also equidistant  
from two parallel lines.

Let  $p$  and  $p'$  be the given points  
and  $ab$  and  $cd$  the given parallel lines.

To find a point  $x$  equidistant  
from  $p$  and  $p'$  and also from  $ab$  and  $cd$ .

From  $a$  draw  $ac$  perpendicular to  
 $cd$ ,

and bisect  $ac$  at  $e$ .

Through  $e$ , draw  $ef$  parallel to  $ab$ .

Construct  $e'f'$  perpendicular to  $pp'$  at its middle point.

Find the intersection,  $x$ , of  $ef$  and  $e'f'$ .

Then  $x$  is the required point.

Proof: Since  $x$  lies on  $ef$  it is equidistant from  $ab$  and  $cd$ .

(W., Art., 181)

and since it lies on  $e'f'$  it is equidistant from  $p$  and  $p'$ . (W., Art., 160)

Therefore  $x$  is the required point.

49. To find a point  $x$  equidistant from  
two given intersecting lines and also equi-  
distant from two given parallels.

Let  $ab$  and  $a'b'$  be the two given parallels and  $cd$  and  $c'd'$  the two  
given intersecting lines.

To find a point  $x$  equidistant from  $ab$  and  $a'b'$  and also equidistant



from  $cd$  and  $c'd'$ .

$+f'$

Find the intersection,  $o$ , of  $cd$  and  $c'd'$ .

$a+$

$+h$

Construct the bisector,  $e'f'$ , of the angle

$e+$

$+x$

$+f$

$coe'$ . (Art., 7) & (Art., 15)

$a'+$

$+h'$

Draw  $ef$  parallel to  $ab$  and midway between

$c+$

$+c'$

$ab$  and  $a'b'$  as in article 48. Find the

$xo$

intersection,  $x$ , of  $ef$  and  $e'f'$ ! (Art., 7)

$d'+$

$+e'$

$+d$

Then  $x$  is the required point.

Proof: Since  $x$  lies on  $ef$  it is equidistant from  $ab$  and  $a'b'$  and since it lies on  $e'f'$  it is equidistant from  $cd$  and  $c'd'$  (W., Arts., 181, 162)

Therefore  $x$  is the required point.

50. To find a point  $x$  equidistant from two given intersecting lines and at a given distance from a given point.

Let  $r$  be the given distance,

$a+$

$+c$

$p$  the given point and  $ab$  and  $cd$

$a+$

$o+$

$-x$

$x'-$

$+f$

the given intersecting lines.

$d+$

$+h$

To find a point  $x$  equidistant

from  $ab$  and  $cd$  and at the distance  $r$  from  $p$ .

Find the intersection,  $c$ , of  $ab$  and  $cd$ . (Art., 7)

Construct the bisector,  $ef$ , of the angle  $cob$ . (Art., 15)

With  $p$  as a centre and  $r$  as a radius, describe a circle and find its intersections,  $x$  and  $x'$ , with  $ef$ . (Art., 4, 41)

Then  $x$  and  $x'$  are the required points.

Proof: Since  $x$  and  $x'$  lie on the circumference of the circle, they are at a distance  $r$  from  $p$  and since they lie on  $ef$  they are equidistant from  $ab$  and  $cd$ . (W., Art., 162)

Therefore  $x$  and  $x'$  are the required points.

51. To find a point  $x$  which lies in one side of a given triangle and is equidistant from the other two sides.

Let  $abc$  be the given triangle.

$d+$

$+a$

To find a point  $x$  in  $ac$  equidistant from  $ab$  and  $bc$ .

Construct the bisector,  $bd$ , of the angle  $abc$ . (Art., 15)

$x x$

Find the intersection,  $x$ , of  $ac$  and  $bd$ .

$c+$

$+h$

(Art., 7)

Then  $x$  is the required point.



Proof: Since  $x$  lies in  $bd$  it is equidistant from  $ab$  and  $bc$ .

(W., Art., 162)

It also lies in  $ac$ . Therefore  $x$  is the required point.

52. A straight railway passes two miles from a town. A place is four miles from the town and one mile from the railway. To find by construction the places that answer this description.

Let  $o$  be the town and  $ab$  the railway.

To find points four miles from  $o$  and one mile from  $ab$ .

Construct  $a'b'$  and  $a''b''$  parallel to  $ab$  and at a distance one from it as in article 48. With  $o$  as a centre and radius four, describe a circle. Find the intersections  $p, p', p'', p'''$  of this circle with  $a'b'$  and  $a''b''$ .

(Art., 4, III)

Then  $p, p', p'', p'''$  are the required points.

Proof: Since these points lie on the circumference of the circle, they are at the distance four from  $o$ , and since they lie on  $a'b'$  and  $a''b''$ , they are at the distance one from  $ab$ .

(W., Art., 181)

Therefore these are the required points.

53. In a triangle  $abc$ , to draw  $de$  parallel to base  $bc$ , cutting the sides of the triangle in  $d$  and  $e$ , so that  $de$  shall equal  $db+ec$ .

Let  $abc$  be the given triangle.

To draw  $de$  parallel to the base so that  $de=db+ec$ .

Construct the bisectors,  $ch$  and  $bg$ , of the angles  $acb$  and  $abc$ . (Art., 15)

Find the intersection,  $f$ , of  $ch$  and  $bg$ . (Art., 7)

Through  $f$  draw  $de$  parallel to  $bc$ .

Find its intersections,  $d$  and  $e$ , with  $ab$  and  $ac$ .

Then  $de$  is the required line..

Proof:  $de$  is parallel to  $bc$ .

Then  $\angle fbc = \angle bfd$ .

But  $\angle fbc = \angle fbd$ .

Hence  $\angle fbd = \angle dfb$ .

and therefore  $df=db$ .

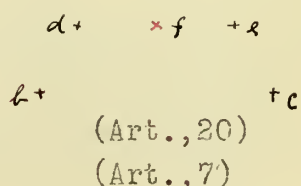
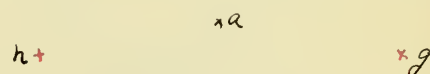
Similarly  $fe=ec$  and therefore  $de=db+ec$ . Therefore  $de$  is the required line.

Const.

(W., Art., 110)

Const.

(W., Art., 147)







54. To draw through two sides of a triangle a line parallel to the third side so that the part intercepted between the sides shall have a given length.

Let  $abc$  be the given triangle, and  $m$  the given length.

To draw a line parallel to  $bc$  so that the part,  $fe$ , intercepted between  $ab$  and  $ac$  shall have the given length  $m$ .

Lay off  $bd$  equal to  $m$ .  
Through  $d$ , draw  $de$  parallel to  $ab$ .  
Find its intersection,  $e$ , with  $ac$ .  
Through  $e$ , draw  $ef$  parallel to  $bc$ .  
Then  $ef$  is the required line.

Proof?  $bdef$  is a parallelogram.  
Hence  $ef = bd = m$ .  
Therefore  $ef$  is the required line.

$++$   $+l$   
 $b+$   $+d +c$   
(Art., 4, Cor.)  
(Art., 20)  
(Art., 7)  
(Art., 20)  
Const.  
(W., Art., 178)

55. Prove that the locus of the vertex of a right triangle, having a given hypotenuse as base, is the circumference described upon the given hypotenuse as diameter.

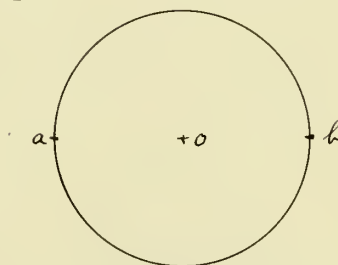
Let  $ab$  be the given hypotenuse.

To prove that the locus of the vertex of the right triangle, having  $ab$  as a hypotenuse is the circumference of a circle whose diameter is  $ab$ .

Proof: Bisect  $ab$  at  $o$ .

Construct a circle with  $o$  as a centre and  $oa$  as a radius. Any triangle found with  $ab$  as one side and its vertex in the circumference is a right triangle.

Therefore the locus of the vertex of a right triangle having  $ab$  as the hypotenuse is the circumference.



(Art., 14)

(W., Art., 290)

56. Prove that the locus of the vertex of a triangle, having a given base and a given angle at the vertex is the arc which forms with the base a segment capable of containing the given angle.

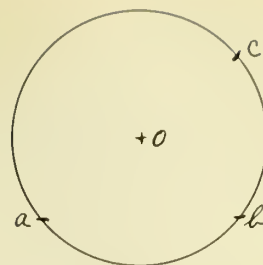
Let  $ab$  be the given base and  $acb$  the given angle.



To prove that the locus of the vertex of a triangle having  $ab$  as a base and  $\angle acb$  as the angle at the vertex is the arc  $acb$ .

Proof: Construct a segment on  $ab$  in which the given angle at the vertex may be inscribed. (Art., 38)

Any triangle having  $ab$  as a base and its vertex in the arc of this segment will have the required angle at the vertex. Hence the arc of this segment is the locus of the vertex of a triangle having  $ab$  as a base and the given angle at its vertex.



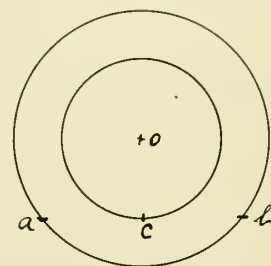
57 Find the locus of the middle point of a chord of a given length that can be drawn in a given circle.

Let  $ab$  be the chord in the circle whose centre is  $o$ .

To find the locus of the middle point of  $ab$ .

Bisect  $ab$  at  $c$ . (Art., 14)

Then since equal chords are equally distant from the centre, the locus of the point  $c$  is the circumference of a circle whose centre is  $o$  and radius  $oc$ .



58. Find the locus of the middle point of a chord drawn from a given point in a given circumference.

Let  $c$  be the middle point of the chord  $ab$  in the circle whose centre is  $o$ .

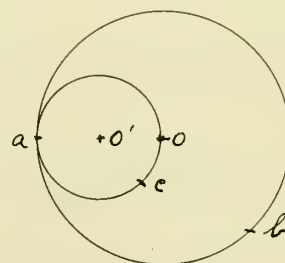
To find the locus of  $c$ .

$oc$  is perpendicular to  $ab$ . (W., Art., 247)

Therefore the locus of the point  $c$  is the locus of the vertex of a right triangle having  $ao$  as a hypotenuse. By article 55 this is a circle having  $ao$  as a diameter. Bisect  $ao$  at  $o'$ .

(Art., 14)

With  $o'$  as a centre and  $o'a$  as a radius, describe a circle. Then this is the required locus.



59. Find the locus of the middle point of a straight line drawn from a given exterior point to a given circumference.

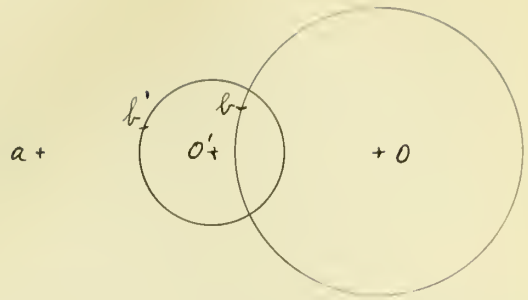
Let  $a$  be the given external point, and  $o$  the centre of the given circumference.



To find the locus of the middle point of a line drawn from  $a$  to the given circumference.

Bisect  $ao$  at  $o'$ . (Art., 14)

With  $o'$  as a centre and a radius equal to one half of  $ob$ , describe a circle. Then this circumference whose centre is  $o'$  is the required locus.



Proof: Take any point,  $b$ , in the circumference whose centre is  $o$  and find the intersection  $b'$  of  $ab$  with the circumference whose centre is  $o'$ .

(Art., 4, II)

Then  $ao:ao'::ob:o'b'::2:1$ .

Const.

Then the triangles  $aob$  and  $ao'b'$  are similar.

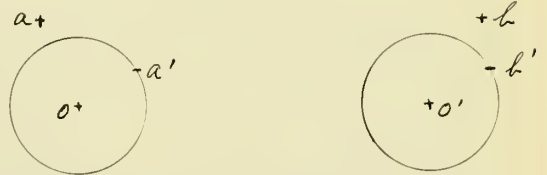
(W., Art., 357)

Hence  $ao:ao'::ab:ab'::2:1$ .

Since  $b$  is any point on the given circumference and  $b'$  is the middle point of  $ab$ , then the circumference whose centre is  $o'$  is the required locus.

60. A straight line moves so that it remains parallel to a given line, and touches at one end a given circumference. Find the locus of the other end.

Let  $ab$  be the given fixed line,  $a'b'$  any position of the movable line and  $o$  the centre of the given circumference.



To find the locus of  $b'$  as  $a'$  moves on the given circumference and  $a'b'$  remains parallel to  $ab$ .

Construct the parallelogram  $oa'b'o'$ .

(Art., 33)

With  $o'$  as a centre and  $oa'$  as a radius, describe a circle. Then this is the required locus.

Proof: Since  $a'b'$  moves parallel to  $ab$  it will always remain parallel and equal to  $oo'$ . Hence  $oa'b'o'$  is always a parallelogram.

(W., Art., 183)

Hence  $o'b'$  is always equal to  $oa'$ .

(W., Art., 178)

Therefore the circumference whose centre is  $o'$  is the required locus.

61. A straight rod moves so that its ends constantly touch two fixed rods which are perpendicular to each other. Find the locus of its middle point.

Let  $ao$  and  $bo$  be the two given lines at right angles,  $cd$  the movable

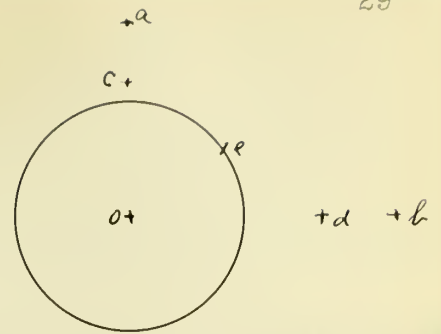




line and  $e$  its middle point.

To find the locus of  $e$  as  $cd$  moves on  $ao$  and  $bo$ .

With  $o$  as a centre and  $oe$  as a radius describe a circle. The part of this circumference included between  $ao$  and  $bo$  is the required locus.



Proof: In every position of  $cd$ , we have  $oe = \frac{1}{2} cd$  for  $oe$  is always the line joining the middle point of the hypotenuse of a right triangle to the vertex of the right angle which is half the hypotenuse. Therefore the part of the circumference between  $ao$  and  $bo$  is the required locus.

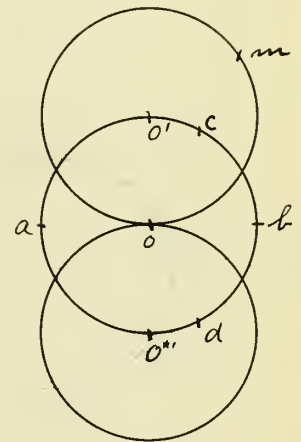
62. In a given circle let  $ab$  be a diameter,  $oc$  any radius,  $cd$  the perpendicular from  $c$  to  $ab$ . Upon  $oc$  take  $om$  equal to  $cd$ . Find the locus of the point  $m$  as  $oc$  turns about  $o$ .

Let  $ab$  be the given diameter,  $oc$  any radius,  $cd$  perpendicular to  $ab$  and  $om = cd$ .

To find the locus of the point  $m$  as  $oc$  turns about  $o$ .

Erect a perpendicular to  $ab$  at  $o$  (Art., 12, I) and find its intersections  $o'$  and  $o''$  with the circle. (Art., 4, I)

With  $o'$  and  $o''$  as centres and  $oa$  as a radius describe circles. Then these circumferences are the required locus.



Proof: In the two triangles  $oo'm$  and  $ocd$ ,  $om = cd$ ,  $oo' = od$  and  $\angle odc = \angle o'm$ . Hence  $\text{tri. } oo'm = \text{tri. } ocd$ . (W., Art., 143)

Hence  $o'm = oc$ . Since  $ocm$  is any position of the radius, then the locus of  $m$  will be the circumference of a circle with  $o'$  as a centre and  $o'm$  as radius. When  $oc$  gets below  $ab$  the locus of  $m$  will be the circumference of a circle having  $o''$  as a centre and  $oa$  as a radius.

63. To construct an equilateral triangle, having given the radius of the circumscribed circle.

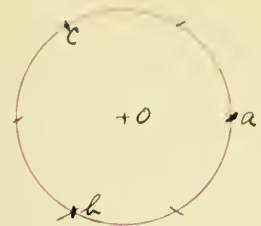
Let  $oa$  be the given radius.

To construct an equilateral triangle the radius of whose circumscribed circle is  $oa$ .

With  $o$  as a centre and  $oa$  as a radius describe a circle. Inscribe



a regular hexagon in this circle. (W., Art., 469)  
Join the alternate vertices a, b, c of this hexagon and we have our required triangle.



64. To construct an isosceles triangle, having given:

I. The angle at the vertex and the base.

Let mn be the given base and abc the given angle at the vertex.

To construct an isosceles triangle having mn for a base and abc for the angle at the vertex.

Extend the line cb to d.

(Art., 4, Cor.)

Bisect the angle abd by be.

(Art., 15)

At m and n construct  $\angle pmn = \angle pnm = \angle abc$ .

(Art., 19)

Find the intersection p of mp and np.

(Art., 7)

Then mnp is the required triangle.

Proof: Since  $\angle m = \angle n$ , the triangle mnp is isosceles. (W., Art., 147)

$$\angle p = 180^\circ - (\angle m + \angle n) = 180^\circ - 2\angle abc = \angle abc$$

Const.

Therefore mnp is the required triangle.

II. The base and the radius of the circumscribed circle.

Let ab be the given base and ao the radius of the circumscribed circle.

To construct an isosceles triangle having ab for a base and ao for the radius of the circumscribed circle.



With a and b as centres and ao as a radius, describe arcs intersecting at o. With o as a centre and oa as a radius, describe a circle. Erect a perpendicular to ab at its middle point.

(Art., 12, I)

Find its intersection, c, with the circumference.

(Art., 4, I)

Then abc is the required triangle.

Proof:  $ac = bc$ .

(W., Art., 160)

Hence triangle abc is isosceles.

(W., Art., 120)

Therefore abc is the required triangle.

III. The base and the radius of the inscribed circle.

Let ab be the given base and od the radius of the inscribed circle.

To construct an isosceles triangle having ab as a base and od as the radius of the inscribed circle.



Erect a perpendicular at the middle point  
of  $ab$ . (Art., 12, I)  
Lay off  $od$  equal to the given radius. (Art., 4, Cor.)  
With  $o$  as a centre and  $od$  as a radius, describe a  
circle.

Draw tangents to this circle from  $a$  and  $b$ .

(Art., 36, II)

Find their intersection  $c$ .

(Art., 7)

Then  $abc$  is the required triangle.

Proof:  $\text{tri. } adc = \text{tri. } bdc$ .

(W., Art., 143)

Then  $ac = bc$  and  $\text{tri. } abc$  is isosceles.

(W., Art., 120)

Therefore  $abc$  is the required triangle.

IV. The perimeter and the altitude.

Let  $ab$  be the given perimeter and  $cd$  the  
given altitude.

To construct an isosceles triangle having  
 $ab$  as a perimeter and  $cd$  as an altitude.

Erect a perpendicular at the midpoint  
of  $ab$ . (Art., 12, I)

Lay off  $cd$  equal to the given altitude.

(Art., 4, Cor.)

Erect perpendiculars at the middle points of  $ac$  and  $bc$ . (Art., 12, I)

Find their intersections,  $e$  and  $f$ , with  $ab$ .

(Art., 7)

Then  $efc$  is the required triangle.

Proof:  $ae = ec$  and  $bf = cf$ .

(W., Art., 160)

Hence  $ec + cf + fe = ab$ .

Triangle  $ecf$  is also isosceles and is therefore the required triangle.

65. To construct a right triangle, having  
given:

I The hypotenuse and one leg.

Let  $ab$  be the hypotenuse and  $ac$  the given  
leg.

To construct a right triangle having  $ab$   
for a hypotenuse and  $ac$  for one leg.

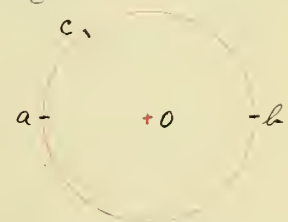
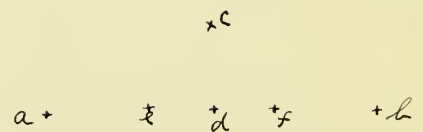
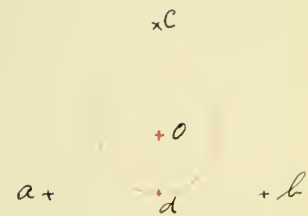
Construct a circle on  $ab$  as a diameter.

With  $a$  as a centre and the given leg as a radius, cut the circumference at  
 $c$ . Then  $abc$  is the required triangle.

Proof:  $acb$  is a right triangle.

(W., Art., 290)

$ab$  is the given hypotenuse and  $ac$  is the given leg. Therefore  $abc$  is the  
required triangle.







II. One leg and the altitude upon the hypotenuse.

Let  $ac$  be the given leg and  $cd$  the given altitude.

To construct a right triangle with  $ac$  as one leg and  $cd$  as the altitude upon the hypotenuse.

At any point in the line  $mn$  construct a perpendicular  $dc$ .

(Art., 12, I)

Lay it off equal to the given altitude.

(Art., 4, Cor.)

With  $c$  as a centre and the given leg as a radius, strike an arc cutting  $mn$  at  $a$ . Erect a perpendicular to  $ac$  at  $c$ .

(Art., 12, I)

Find its intersection,  $b$ , with  $mn$ .

(Art., 7)

Then  $abc$  is the required triangle.

Proof:  $acb$  is a right angle.

Const.

$ac$  is the required leg and  $cd$  is the required altitude. Therefore  $abc$  is the required triangle.

III. The median and the altitude drawn from the vertex of the right angle.

Let  $oc$  be the median and  $od$  the altitude.

To construct a right triangle with  $oc$  as the median and  $od$  as the altitude drawn from the vertex of the right angle.

$c - +d - c'$

$a - +o - b$

Construct a circle with the given median as a radius and to any diameter,  $ab$ , construct a perpendicular,  $od$ , at its middle point.

(Art., 12, I)

Lay it off equal to the given altitude.

(Art., 4, Cor.)

Draw a line through  $d$  parallel to  $ab$

(Art., 20)

and find its intersection,  $c$  and  $c'$ , with the circle.

(Art., 4, II)

Then  $abc$  and  $abc'$  are the required triangles.

Proof:  $od$  is the required altitude and  $oc$  the required median. Const. Angles  $acb$  and  $ac'b$  are right angles.

(W., Art., 290)

Therefore  $acb$  and  $ac'b$  are the required triangles.

IV. The hypotenuse and the altitude upon the hypotenuse.

Let  $ab$  be the given hypotenuse and  $od$  the given altitude.

To construct a right triangle with  $ab$  for a hypotenuse and  $od$  for an altitude.

Construct a circle on  $ab$  as a diameter, erect a perpendicular at the middle point

$c - +d - c'$

$a - +o - b$



of ab, and lay it off equal to the altitude od. (Arts., 12, I; 4, Cor.)  
 Through d, draw cc' parallel to ab. (Art., 20)  
 Find its intersections c and c' with the circle. (Art., 4, II)  
 Then acb and ac'b are the required triangles.

Proof: They have the given hypotenuse and altitude. Const.  
 Angles acb and ac'b are right angles. (W., Art., 290)  
 Therefore acb and ac'b are the required triangles.

V. The radius of the inscribed circle and one leg.

Let ab be the given leg and oe the given radius.

To construct a right triangle, having ab as one leg and oe as the radius of the inscribed circle.

Draw mn parallel to ab and at a distance oe from it, (Art., 20).  
 Construct  $\angle abd = 45^\circ$ . (Art., 16)  
 Find the intersection o of mn and bd. (Art., 7)  
 With o as a centre and oe as a radius, describe a circle.  
 Draw tangents to this circle from a and b and find their intersection c. (Art., 36, II)

Then abc is the required triangle.

Proof: abc is circumscribed about a circle of the given radius. Const.  
 Since  $\angle abd = 45^\circ$  then  $\angle abc = 90^\circ$ . (W., Art., 261)  
 Therefore abc is the required triangle.

VI. The radius of the inscribed circle and an acute angle.

Let nmp be the given angle and oc the radius of the inscribed circle.

To construct a right triangle having nmp as one of the acute angles and oc as the radius of the inscribed circle.

Draw the bisector md of the angle nmp, (Art., 15)  
 Draw m'n' parallel to mn and at the distance oc from it. (Art., 20)  
 Find the intersection o of m'n' and md. (Art., 7)  
 With o as a centre and the given radius, describe a circle and find the point e where it cuts m'n'. (Art., 4, I)  
 Draw a tangent to the circle at e. (Art., 36, I)  
 Find its intersection a and b with mn and mp. (Art., 7)  
 Then mab is the required triangle.

Proof: By construction, mab is circumscribed about the given circle and also has the given acute angle.



ab is perpendicular to m'n'

(W., Art., 254)

and is therefore perpendicular to mn.

(W., Art., 197)

Therefore mab is the required right triangle.

VII. An acute angle and the sum of the legs.

Let ae be the sum of the legs and ead the given acute angle.

To construct a right triangle with dae as an acute angle and ae as the sum of the legs.

a+ b+ c

At e construct  $\angle aec = 45^\circ$ . (Art., 16)

Find the intersection, c, of ad and ec.

(Art., 7)

From c, draw cb perpendicular to ae.

(Art., 12, II)

Then abc is the required triangle.

Proof: By construction, abc is a right triangle having the given acute angle bac.

Also bc=be and hence ab+bc=ab+be.

(W., Art., 147)

Therefore abc is the required triangle.

VIII. An acute angle and the difference of the legs.

Let dae be the given acute angle and ae as the difference of the legs.

+d

Construct  $\angle fec = 45^\circ$ . (Art., 16)

c

Find the intersection, c, of ad and ec.

a+ b+ c

(Art., 7)

From c draw cb perpendicular to af.

(Art., 12, II)

Then abc is the required triangle.

Proof: By construction, abc is a right triangle having the given acute angle dae.

Also be=bc and hence ae=ab-bc.

(W., Art., 147)

Therefore abc is the required triangle.

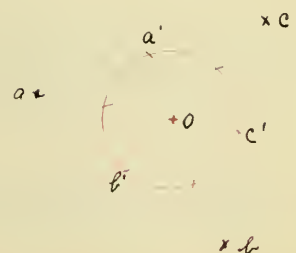
66. To construct an equilateral triangle having given the radius of the inscribed circle.

Let oa' be the given radius.

To construct an equilateral triangle of which oa' is the radius of the inscribed circle.

Describe a circle with oa' as a radius and in it inscribe an equilateral triangle a'b'c'.

(Art., 63)







Draw tangents to the circle at a', b' and c' (Art., 26, I)  
 and find their intersections, a, b and c. (Art., 7)  
 Then abc is the required triangle.

Proof: abc is Circumscribed about the circle. Const.  
 abc is an equilateral triangle. (W., Art., 440)  
 Therefore abc is the required triangle.

67. To construct a triangle having;

I. The base, the altitude, and the angle at the base.

Let ab be the given base, aa' the altitude and bad the given angle.

To construct a triangle having these given parts.

Draw a'b' parallel to ab and at a distance aa' from it, (Art., 20)  
 and find the intersection c of ad and a'b'. (Art., 7)

Then abc is the required triangle.

Proof: The triangle abc has the required base ab and the required angle bad. Const.

Its altitude is equal to the given altitude aa'. (W., Art., 181)  
 Therefore abc is the required triangle.

II. The base, the altitude, and the angle at the vertex.

Let ab be the given base, cd the given altitude and acb the given angle at the vertex.

To construct a triangle having these given parts.

Upon ab describe a segment in which the given angle acb may be inscribed. (Art., 38)

Draw a'b' parallel to ab and at a distance cd from it. (Art., 20)

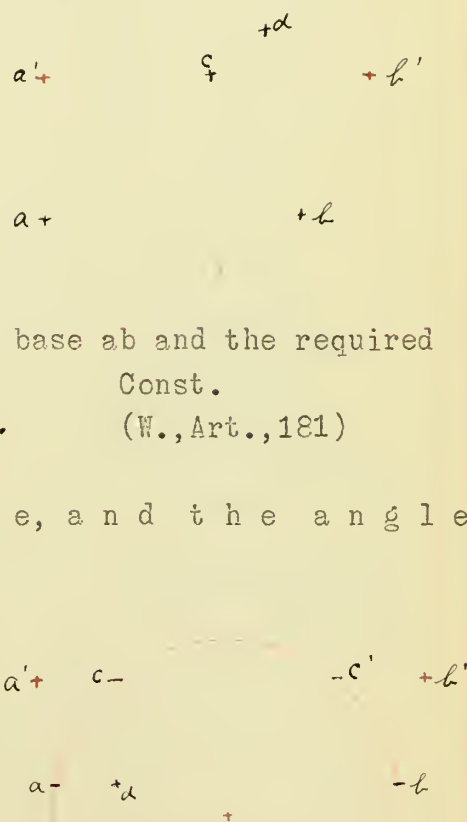
Find intersections c and c' with the arc.

Then abc or abc' is the required triangle.

Proof; ab is the given base, cd the given altitude and acb or ac'b the given angle at the vertex. Therefore abc is the required triangle.

III. The base, the corresponding median and the angle at the vertex.

Let ab be the given base, oc the median and acb the given angle at the vertex.





To construct a triangle having these given parts.

Upon  $ab$  describe a segment in which the given angle  $acb$  may be inscribed. (Art., 38)

Bisect  $ab$  at  $o$ . (Art., 14)

With  $o$  as a centre and the given median as a radius strike an arc locating  $c$  and  $c'$ .

Then  $abc$  or  $abc'$  is the required triangle.

Proof: By construction  $ab$  is the given base,  $oc$  the given median and  $acb$  the given angle at the vertex, Therefore  $abc$  or  $abc'$  is the required triangle.

#### IV. The perimeter and the angles.

Let  $mn$  be the given perimeter and  $\phi, \psi$  and  $\theta$  the given angles.

To construct a triangle whose perimeter is equal to  $mn$  and whose angles are equal to  $\phi, \psi$  and  $\theta$ .

Construct  $\angle mne = \phi/2$  and  $\angle mnd = \psi/2$ .

(Art., 19)

Find the intersection  $c$  of  $me$  and  $nd$

Construct  $\angle mca = \phi/2$  and  $\angle ncb = \psi/2$

Find the intersections  $a$  and  $b$  with  $mn$

Then  $abc$  is the required triangle.

Proof:  $ma = ac$  and  $nb = bc$ .

Therefore  $ab + ac + bc = mn$ .

Also  $\angle bac = \phi/2 + \phi/2 = \phi$  and  $\angle abc = \psi/2 + \psi/2 = \psi$ .

Therefore  $\angle bca = \theta$ .

Therefore  $abc$  is the required triangle.

V. One side, an adjacent angle and the sum of the other sides.

Let  $ab$  be the given side,  $bac$  the given angle and  $ad$  the sum of the other two sides.

To construct a triangle having  $ab$  as one side,  $bac$  as one angle and  $ad$  as the sum of the other two sides.

Erect a perpendicular  $ef$  at the middle point of  $bd$ . (Art., 12, I)

Find the intersection  $c$  of  $ad$  and  $ef$ .

Then  $abc$  is the required triangle.

Proof:  $bc = cd$ .

Hence  $ac + cb = ac + cd$ .

Therefore  $abc$  is the required triangle.



VI. One side, an adjacent angle and the difference of the other two sides.

Let  $ab$  be the given side,  $\angle bae$  the given angle and  $ad$  as the difference of the other two sides.

Construct  $fg$  perpendicular to  $bd$  at its middle point. (Art., 12, I)  
Find the intersection  $c$  of  $ae$  and  $fg$ .  
Then  $abc$  is the required triangle.

Proof:  $bc=dc$ .

Hence  $ad=ac-bc$

Therefore  $abc$  is the required triangle.

VII. The sum of the two sides and the angles.

Let  $af$  be the sum of the two given sides and  $\phi, \psi$  and  $\theta$  the given angles.

To construct a triangle whose angles are  $\phi, \psi$  and  $\theta$  and the sum of two of whose sides is equal to  $af$ .

Construct  $\angle fad = \phi$  and  $\angle afe = \psi$ .

(Art., 19)

Find the intersection  $e$  of  $fe$  and  $ad$ .

Bisect  $\angle afe$  by  $fc$  and find its intersection  $c$  with  $ad$ .

Through  $c$ , draw  $cb$  parallel to  $ef$ .

Find the intersection  $b$  of  $cb$  and  $af$ .

Then  $abc$  is the required triangle.

Proof;  $\angle bcf = \angle cfe = \angle bfc = \angle \psi / 2$ .

Hence  $bc=bf$  and  $ab+bc=ab+bf=af$ ,

also  $\angle abc = \angle afe = \angle \psi$ ,

and  $\angle bac = \phi$ ,

then  $\angle acb = \theta$ .

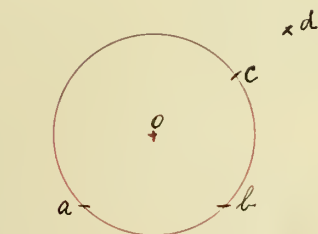
Therefore  $abc$  is the required triangle.

VIII. One side, an adjacent angle and the radius of the circumscribed circle.

Let  $ab$  be the given side,  $\angle bad$  the given angle, and  $oa$  the radius of the circumscribed circle.

To construct a triangle having these given parts.

With  $a$  and  $b$  as centres and  $oa$  as a radius, describe arcs intersecting at  $o$ . With  $o$  as a centre and  $oa$  as a radius, describe a circle







and find its intersection  $c$  with  $ad$ .

(Art., 4, II)

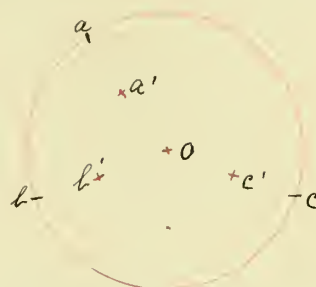
Then  $abc$  is the required triangle.

Proof: By construction it has the given side  $ab$ , the given angle  $bac$  and the given circumscribed circle. Therefore it is the required triangle.

IX. The angles and the radius of the circumscribed circle.

Let  $a', b'$  and  $c'$  be the given angles and  $oa$  the radius of the circumscribed circle.

To construct a triangle having these given angles and inscribed in the given circle.



Construct the triangles  $a'b'c'$  having the given angles and find the centre  $o$  of its circumscribed circle.

(Art., 34)

With  $o$  as a centre and  $oa$  as a radius, describe a circle.

Find the intersections  $a, b$  and  $c$  of  $oa', ob'$  and  $oc'$  with the circumference.

(Art., 4, I)

Then  $abc$  is the required triangle.

Proof:  $abc$  is inscribed in the given circle.

Const.

$ab$  is parallel to  $a'b'$ ,  $bc$  to  $b'c'$  and  $ac$  to  $a'c'$ .

(W., Art., 3459)

Hence  $\angle a = \angle a'$ ,  $\angle b = \angle b'$  and  $\angle c = \angle c'$ .

(W., Art., 176)

Therefore  $abc$  is the required triangle.

X. The angles and the radius of the inscribed circle.

Let  $a', b'$  and  $c'$  be the given angles and  $op$  the radius of the inscribed circle.

To construct a triangle having these given angles and circumscribed about the given circle.

Construct any triangle  $a'b'c'$  having the given angles and find the centre  $o$  of its inscribed circle.

(Art., 35)

With  $o$  as a centre and  $op$  as a radius, describe a circle.

Draw  $op$  perpendicular to  $a'b'$ ,  $op'$  to  $a'c'$  and  $op''$  to  $b'c'$ .

(Art., 12, II)

Find the intersections  $p, p'$  and  $p''$ .

(Art., 36, I)

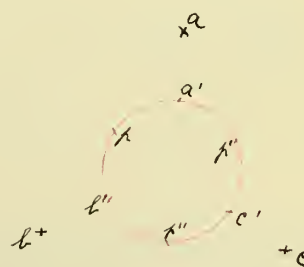
Draw tangents to the circle at  $p, p'$  and  $p''$ .

(Art., 36, I)

Find their intersections  $a, b$  and  $c$ .

(Art., 7)

Then  $abc$  is the required triangle



Proof:  $abc$  is circumscribed about the circle.

Const.

$op$  is perpendicular to  $ab$ ,  $op'$  to  $ac$  and  $op''$  to  $bc$ .

(W., Art., 254)

Then  $ab$  is parallel to  $a'b'$ ,  $ac$  to  $a'c'$ ,  $bc$  to  $b'c'$ .

(W., Art., 104)

Hence  $\angle a = \angle a'$ ,  $\angle b = \angle b'$  and  $\angle c = \angle c'$ .

(W., Art., 176)



Therefore  $abc$  is the required triangle.

XI. An angle, and the bisector and the altitude drawn from the vertex of the given angle.

Let  $ead$  be the given angle,  $ag$  its bisector and  $ao$  the given altitude.

To construct a triangle having these given parts.

Construct the bisector  $ag$  of the angle  $ead$ . (Art., 15)

Lay it off equal to the given bisector.

(Art., 4, Cor.)

With  $a$  as a centre and the given altitude as a radius, describe a circle.

From  $g$  draw tangents  $gb$  and  $gc'$  to the circle. (Art., 36, II)

and find the intersections  $b, c, b'$  and  $c'$  of these tangents with  $ae$  and  $ad$ .

(Art., 7)

Then  $abc$  or  $ab'c'$  is the required triangle.

Proof: By construction, either of the triangles  $abc$  or  $ab'c'$  has the given angle and the given bisector. The altitude is the perpendicular from  $a$  to  $bc$  or  $b'c'$  which in either case is  $ao$ , the given altitude.

(W., Art., 253)

Therefore  $abc$  or  $ab'c'$  is the required triangle.

XII. Two sides and the median corresponding to the other side.

Let  $ab$  and  $ac$  be the two given sides and  $ad$  the given median.

To construct a triangle having  $ab$  and  $ac$  as two of its sides and  $ad$  the median to the third side.

Bisect  $ab$  at  $f$ . (Art., 15)

With  $f$  and  $a$  as centres and radii  $1/2 ac$  and  $ad$  respectively, strike arcs intersecting at  $d$ . With  $d$  and  $a$  as centres and radii  $af$  and  $df$  respectively, strike arcs intersecting at  $e$ . Multiply  $ae$  by two, locating the point  $c$ .

(Art., 21)

Then  $abc$  is the required triangle.

Proof: The triangle  $abc$  has the given sides  $ab$  and  $ac$ . Construct the quadrilateral  $aedf$  is a parallelogram. (W., Art., 182)

Then  $ed$  is parallel to  $ab$  and equal to one-half of it.

By construction  $e$  is the middle point of  $ac$ .

Hence  $d$  is the middle point of  $bc$ .

(W., Art., 188)

Therefore  $abc$  is the required triangle.



### XIII. The three medians.

Let  $ad$ ,  $be$  and  $cb$  be the given medians.

To construct a triangle having these given medians.

Divide  $ad$  into three equal parts at  $o$  and  $g$ . (Art., 22)

With  $g$  and  $d$  as centres and radii equal respectively to  $1/3 be$  and  $1/3 ch$ , describe arcs locating the point  $f$ . Extend  $fg$  to  $e$  and  $b$  making  $eg = fb = 1/3 eb$ . (Art., 4, Cor.)

Extend the lines  $ae$  and  $bd$ .

Find their intersection  $c$ .

Then  $abc$  is the required triangle.

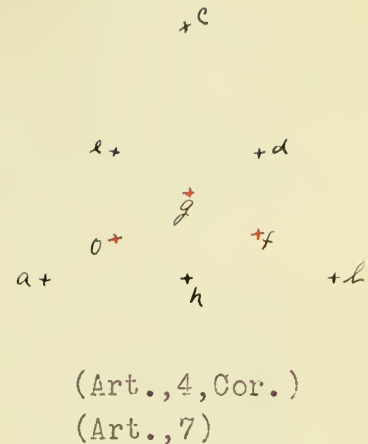
Proof:  $ed = of = 1/2 ab$  and the lines are parallel. (W., Art., 189)

Hence  $edfo$  is a parallelogram.

(W., Art., 183)

Find the intersections  $h$  and  $r$  of the median from  $c$  with  $ab$  and  $of$ ,

Then  $gr = 1/2 df = 1/2 gh$ . Hence  $df = gh$  and since  $df = 1/3 ch$  we have  $gh = 1/3 ch$  and therefore  $ch$  is the given median. Therefore the triangle has the three given medians and must be the required triangle.



### 68. To construct a square, having given;

#### I. The diagonal.

Let  $ac$  be the given diagonal.

To construct a square having  $ac$  as a diagonal.

Construct a circle on  $ac$  as a diameter.

(Art., 14)

Erect a perpendicular to  $ac$  at  $o$ . (Art., 12, I)

Find its intersection  $b$  and  $d$  with the circumference.

(Art., 4, I)

Then  $abc$  is the required square.

Proof: Since the angles at  $o$  were constructed equal, then the arcs  $ab$ ,  $bc$ ,  $cd$  and  $da$  are equal and therefore the chords  $ab$ ,  $bc$ ,  $cd$  and  $da$  are equal.

(W., Art., 236, 241)

Also  $\angle a = \angle b = \angle c = \angle d = a$  right angle.

(W., Art., 290)

Therefore  $abcd$  is the required square.

#### II. The sum of the diagonal and one side.

Let  $fc$  be the sum of the diagonal and one side.

To construct a square such that the sum of its side and diagonal shall be equal to  $fc$ .



$f + a + c$

$d$   
 $l$





At c construct  $\angle fce = 45^\circ$ , (Art., 16)  
 and at f construct  $\angle cfd = \frac{1}{2} \angle fce$ . (Arts., 15, 19)  
 Find the intersection, d, of fd and ce. (Art., 7)  
 With d as a centre and dc as a radius, describe an arc cutting fc at a. (Art., 4, II)

With a and c as centres and the same radius, describe arcs intersecting at b. Then abcd is the required square.

Proof:  $ab = bc = cd = da$ . Const.  
 Since  $ad = dc$  then  $\angle dac = \angle acd = 45^\circ$  (W., Art., 145)  
 Therefore  $\angle adf = 2 \times 45^\circ = 90^\circ$ . (W., Art., 137)  
 Therefore  $af = ad$ . (W., Art., 147)  
 Since  $\angle dac = \angle acd = 45^\circ$  then  $\angle adc = 90^\circ$  (W., Art., 129)  
 Similarly for the other angles. Therefore abcd is the required square.

69. Given two perpendiculars, ab and cd, intersecting in o, and a straight line intersecting these perpendiculars in e and f; to construct a square, one of whose angles shall coincide with one of the right angles at o, and the vertex of the opposite angle of the square shall lie in ef. (Two solutions)

Bisect the angles boc and bod (Art., 15)  
 and find the intersections g and h of these bisectors with ef. (Art., 7)  
 From g and h construct perpendiculars to ab and cd. (Art., 12, MI)  
 And find their intersections n, m, p and q with these lines. (Art., 7)

Then mn or pq is the required square.

Proof: Each square has a vertex at o and one in ef. Const.  
 The angles of nm and pq are right angles, Const.  
 and also  $ng = mg$  and  $ph = gh$ . (W., Art., 162)  
 Therefore nm and pq are squares and either would be the required square.

70. To construct a rectangle, having given;

I, One side and the angle between the diagonals.

Let ab be the given side and eaf the angle between the diagonals.

To construct a rectangle having ab as one side and eaf the angle between the diagonals.



Extend ea to b making ab equal to the given side. (Art., 4, Cor.)

Construct the bisector ag of the angle baf.

(Art., 15)

At b, construct bc perpendicular to ab.

(Art., 12, II)

Find the intersection c of ag and bc.

(Art., 7)

Construct ad perpendicular to ab and cd perpendicular to bc and find their intersection d. Then abcd is the required rectangle.

Proof: By construction the angles are right angles and therefore  $\triangle abcd$  is a rectangle.

The angle between the diagonals  $= 180^\circ - (\angle cab + \angle abd)$  (W., Art., 129)

or  $= 180^\circ - 2\angle cob = 180^\circ - \angle fab = \angle eaf$ .

Therefore abcd is the required rectangle.

II. The perimeter and the diagonal.

Let am be the given perimeter and ac the given diagonal.

To construct a rectangle with the perimeter equal to am and diagonal ac.

Bisect am at n. (Art., 14)

Construct angle anp  $= 45^\circ$ . (Art., 16)

With a as a centre and ac as a radius, describe a circle and find its intersection c with np.

(Art., 4, II)

Construct cb perpendicular to an, cd to cb and ad to ab

and find the intersections b and d.

(Art., 12, 7)

Then abcd is the required rectangle.

Proof: By construction it has the given diagonal ac and since the angles are all right angles it is a rectangle.

Angle bcn  $= 45^\circ$ .

(W., Art., 129)

Therefore bc = bn.

(W., Art., 147)

Therefore  $ab + bc = an = nm$ . Hence  $ab + bc + cd + ad = am$ .

Therefore abcd is the required rectangle.

III. The perimeter and the angle between the diagonals.

Let am be the given perimeter and  $\varphi$  as the angle between the diagonals.

Bisect am at n. (Art., 14)

Construct angle anp  $= 45^\circ$ . (Art., 16)

At a construct angle cab  $= \frac{1}{2}(180^\circ - \varphi)$ .

(Art., 19)



Find the intersection  $c$  of  $ac$  and  $np$ . (Art., 7)  
 Construct  $ad$  and  $bc$  perpendicular to  $ab$  and  $cd$  perpendicular to  $bc$ ,  
 and find the intersections  $b$  and  $d$ . (Arts., 12, 7)  
 Then  $abcd$  is the required rectangle.

Proof:  $abcd$  is a rectangle and has the given perimeter by the same proof as the last article.

The angle,  $\varphi$ , between the diagonals  $= 180^\circ - (\angle cab + \angle abd)$ . (W., Art., 129)

Therefore  $\varphi = 180^\circ - 2[1/2(180^\circ - \varphi)] = \varphi$

Therefore  $abcd$  is the required rectangle.

IV. The difference between two adjacent sides and the angle between the diagonals.

Let  $ae$  be the difference between two adjacent sides and  $\varphi$  the angle between the diagonals.

$$\begin{array}{rcccl} & & & & +f \\ d+ & & & +c & \\ & & & & \\ a+ & & +e & +b & \end{array}$$

To construct a rectangle having these given parts.

At  $a$  construct  $\angle fab = 1/2(180^\circ - \varphi)$ . (Art., 19)

At  $e$  construct  $\angle ceb = 45^\circ$ . (Art., 16)

Find the intersection  $c$  of  $af$  and  $ec$ . (Art., 7)

Construct  $cb$  and  $ad$  perpendicular to  $ab$  and  $cd$  perpendicular to  $bc$  and find the intersections  $b$  and  $d$ . (Arts., 12, 7)

Then  $abcd$  is the required rectangle.

Proof:  $abcd$  is a rectangle and has the required angle between the diagonals by the same proof as in the last article.

$\angle bce = \angle bec = 45^\circ$ . (W., Art., 129)

Therefore  $be = bc$ . (W., Art., 147)

Hence  $ae = ab - bc$ .

Therefore  $abcd$  is the required rectangle.

71. To construct a rhombus, having given;

I. The two diagonals.

$+b$

Let  $ac$  and  $bd$  be the given diagonals.

To construct a rhombus with diagonals  $ac$  and  $bd$ .

$a+ \quad +o \quad +c$

Construct a perpendicular at the middle point of the diagonal  $ac$ . (Art., 12, I)

$+a$

Lay off  $ob = od = 1/2$  the given diagonal  $bd$ .

(Art., 4, I)

Then  $abcd$  is the required rhombus.

Proof: The quadrilateral has the given diagonals and also  $ab = bc = cd = ad$ . Const.

(W., Art., 160)

Therefore  $abcd$  is the required rhombus.





II. One side and the radius of the inscribed circle.

Let  $ab$  be the given side and  $ae$  the radius of the inscribed circle.

To construct a rhombus having these given parts.

Construct a semicircle on  $ab$  as a diameter. (Art., 14)

Construct  $ef$  parallel to  $ab$  and at a distance  $ae$  from it. (Art., 20)

Find the intersections  $g$  and  $h$  of  $ef$  with the circumference. (Art., 4, 11)

With either  $g$  or  $h$  as a centre and  $ae$  as a radius, describe a circle.

From  $a$  and  $b$  draw tangents to the circle and lay them off equal to  $ab$ .

(Arts., 36;  
4, Cor.)

Then  $abcd$  is the required rhombus.

Proof: It has the given side and inscribed circle. Const.

Since the diagonals  $ac$  and  $bd$  bisect each other at right angles,

(W., Art., 290)

then  $ab=ad=bc=dc$ .

(W., Art., 160)

Therefore  $abcd$  is a rhombus and is the required rhombus.

III. One angle and the radius of the inscribed circle.

Let  $gah$  be the given angle and  $ae$  the radius of the inscribed circle.

To construct a rhombus having  $gah$  as one angle and  $ae$  as the radius of the inscribed circle.

Construct the bisector  $ac$  of the angle  $gah$ . (Art., 15)

Construct  $ef$  parallel to  $ab$  and at a distance  $ae$  from it. (Art., 20)

Find the intersection  $o$  of  $ef$  and  $ac$ . (Art., 7)

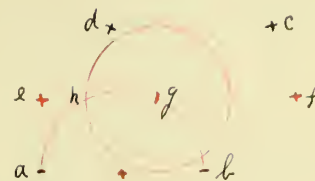
Lay off  $oc=oa$ . (Art., 4, Cor.)

Construct  $cd$  parallel to  $ag$  and  $cb$  parallel to  $ah$  and find the intersections  $d$  and  $b$  with  $gh$  and  $ag$ .

(Art., 20, 7)

Then  $abcd$  is the required rhombus.

Proof: By construction  $abcd$  is a parallelogram and since the diagonal  $ac$  bisects the angles (which are not right angles) it must be a rhombus. Since  $o$  is the middle of the diagonal and at a distance  $ae$  from  $ab$  it is the same distance from all the sides. Therefore this rhombus has the given angle and inscribed circle and must be the required rhombus.





## IV. One angle and one of the diagonals.

Let  $\angle eaf$  be the given angle and  $ac$  the given diagonal.

To construct a rhombus having  $\angle eaf$  as an angle and  $ac$  as a diagonal.

Construct the bisector of the angle  $\angle eaf$  and on it lay off the given diagonal  $ac$ .

(Arts., 15; 4, Cor.)

Construct  $cd$  parallel to  $of$  and  $cb$  parallel to  $ae$  and find their intersections  $d$  and  $b$  with  $ae$  and  $af$ . (Art., 20, 7)  
Then  $abcd$  is the required rhombus.

Proof; By construction,  $abcd$  is a parallelogram and since the diagonal  $ac$  bisects the angles it must be a rhombus. And since it has the given angle and diagonal it must be the required rhombus.

## 72. To construct a rhomboid, having given:

## I. One side and the two diagonals.

Let  $ab$  be the given side and  $ac$  and  $bd$  the given diagonals.

To construct a rhomboid with  $ab$  as a side and  $ac$  and  $bd$  as diagonals.

Bisect the diagonals  $ac$  and  $bd$ . (Art., 14)  
With  $a$  and  $b$  as centres and these half diagonals as radii, describe arcs intersecting at  $o$ .  
Extend  $bo$  to  $d$  and  $ao$  to  $c$  making  $od=ob$  and  $oc=ao$ .  
(Art., 4, Cor.)

Then  $abcd$  is the required rhomboid.

Proof: From the four triangles whose vertices are at  $o$ , we have  
 $dc=ab$  and  $ad=bc$ . (W., Art., 143)  
Hence  $abcd$  is a parallelogram. (W., Art., 182)  
Since the angles are not right angles, it is a rhomboid and must be the required rhomboid.

## II. The diagonals and the angle between them.

Let  $ac$  and  $bd$  be the given diagonals and  $\angle aob$  the given angle.

To construct a rhomboid having  $ac$  and  $bd$  as diagonals and  $\angle aob$  the angle between the diagonals.

Construct  $\angle a'ob'$  equal to the given



angle.

(Art., 19)

Extend a'o to c' and b'o to d'.

(Art., 4, Cor.)

Bisect the given diagonals ac and bd.

(Art., 14)

Lay off  $od=ob=1/2\ bd$  and  $oa=oc=1/2\ ac$ .

(Art., 4, Cor.)

Then abcd is the required rhomboid.

Proof: The quadrilateral abcd has the given diagonals and the angle between them. Const.

From the triangles as in the last article we have that abcd is a rhomboid. Therefore it is the required rhomboid.

III. One side, one angle, and one diagonal.

Let ab be the given side, eab the given angle and ac the given diagonal.

$e + \quad + f$   
 $a + \quad + c$

To construct a rhomboid having these given parts.

Construct bf parallel to ae.

(Art., 20)

$a + \quad + b$

With a as a centre and the given diagonal

as a radius, describe an arc and find its intersection c with bf. (Art., 4II)

Construct cd parallel to ab

and find its intersection d with ae.

(Arts., 20, 7)

Then abcd is the required rhomboid.

Proof: The quadrilateral has the given parts.

Const.

ab is parallel to dc and bc is parallel to ad.

Const.

Therefore abcd is a rhomboid.

(W., Art., 169)

And must be the required rhomboid.

IV. The base, the altitude, and one angle.

Let ab be the given base, ae the altitude and gab the given angle.

$e + \quad + f$   
 $a + \quad + d$   
 $c + \quad + x$

To construct a rhomboid having these given parts.

Through e, construct ef parallel to ab

$a + \quad + b$

and find the intersection d of ag and ef.

(Arts., 29, 7)

Construct a parallelogram abcd having the two sides ab and ad and the included angle bad.

(Art., 33)

Then abcd is the required rhomboid.

Proof: abcd is a parallelogram having the given parts, Const.

and since the angles are not right angles it is the required rhomboid.

(W., Art., 169)

73. To construct an isosceles trapezoid having given :-





## I. The bases and one angle.

Let  $ab$  and  $bh$  be the given bases and  $gab$  the given angle.

To construct a trapezoid having  $ab$  and  $bh$  as bases and  $gab$  as one of the angles.

Construct a perpendicular,  $ef$ , at the middle point of  $ah$ . (Art., 12, I)

Find the intersection,  $d$ , of  $ag$  and  $ef$ .

Construct  $dc$  parallel to  $ab$

and lay it off equal to  $bh$ .

Then  $abcd$  is the required trapezoid.

Proof: By construction  $ab$  and  $cd$  are parallel, therefore  $abcd$  is a trapezoid. (W., Art., 165)

Since  $ad=dh$

(W., Art., 160)

and also  $dh=cb$

(W., Art., 178)

then  $ad=cb$  and therefore  $abcd$  is the required trapezoid.

## II. The bases and the altitude.

Let  $ab$  and  $bh$  be the given bases and  $df$  the given altitude.

To construct an isosceles trapezoid, having the given bases and altitude.

Construct a perpendicular,  $ef$  at the middle point of  $ah$ . (Art., 12, I)

Lay off  $fd$  equal to the given altitude.

Construct  $dc$  parallel to  $ab$

and lay it off equal to  $bh$ .

Then  $abcd$  is the required trapezoid.

Proof: By construction, it is a trapezoid having the given bases and altitude.

Since  $ad=dh$

(W., Art., 160)

and also  $dh=cb$

(W., Art., 178)

then  $ad=bc$  and therefore the trapezoid is isosceles and is the required trapezoid.

## III. The bases and the diagonal.

Let  $ab$  and  $bh$  be the given bases and  $bd$  the given diagonal

To construct an isosceles trapezoid having  $ab$  and  $bh$  as bases and  $bd$  as a diagonal.

Construct a perpendicular,  $ef$ , at the middle point of  $ah$ . (Art., 12, I)

With  $b$  as a centre and  $bd$  as a radius,

$e+g$

$d+ c$

$a+ f+ h+ b$

(Art., 7)

(Arts., 20; 4, Cor.)

$+d$

$d+ c$

$a+ f+ h+ b$

(Art., 4, Cor.)

(Arts., 20; 4, Cor.)

$+d$

$d+ c$

$a+ f+ h+ b$







given parts.

Construct the line  $ef = \frac{1}{2}(ab+cd)$ . (Art., 5, 14)

With e and f as centres and radii  $\frac{1}{2}bd$  and  $\frac{1}{2}ac$  respectively locate the point g. With e and f as centres and radii  $\frac{1}{2}ac$  and  $\frac{1}{2}bd$  respectively locate the point h. Then egfh is a parallelogram. (W., Art., 182)

Through g and h, construct ab and cd respectively parallel to ef and make  $ag=gb=\frac{1}{2}ab$  and  $dh=ch=\frac{1}{2}cd$ . (Art., 20; 4, Cor.)

Then abcd is the required trapezoid.

Proof: By construction, abcd is a trapezoid having the given bases. o is the middle point of hg. (W., Art., 184)

of is parallel to gb and equal to  $\frac{1}{2}(hc+gb)$ . Const.

Therefore f is the middle point of bc. (W., Art., 188)

Hence  $gf = \frac{1}{2}ac$ . (W., Art., 189)

Since gf was constructed as half of the given diagonal, then ac must be the given diagonal. Similarly bd is the other given diagonal. Therefore abcd is the required trapezoid.

III. The bases, one diagonal, and the angle formed by the diagonals.

Let ab and cd be the given bases, ac the diagonal and  $\phi$  the angle between the diagonals.

To construct a trapezoid having these given parts.

Construct the line  $ef = \frac{1}{2}(ab+cd)$ . (Arts., 5, 14)

On the line ef construct the segment of a circle in which the angle  $\phi$  may be inscribed. (Art., 38)

With f as a centre and  $\frac{1}{2}ac$  as a radius, describe an arc cutting the arc of the segment at g.

Construct the parallelogram egfh. (Art., 33)

The construction from here on is the same as in the last article.

Then abcd is the required trapezoid.

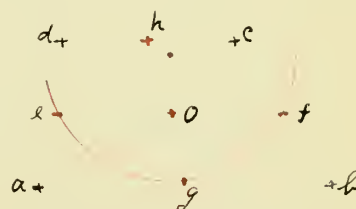
Proof: Since eg is parallel to bd and gf parallel to ac then the angle between the diagonals is equal to the angle  $egf = \phi$ . The trapezoid abcd also has the given bases and diagonal and is therefore the required trapezoid.

75. To construct a circle which has the radius r and which also;

I. Touches each of two intersecting lines ab and cd.

Let ab and cd be the given lines and r the given radius.

To construct a circle of radius r







tangent to  $ab$  and  $cd$ .

Find the intersection  $o$  of  $ab$  and  $cd$ .

(Art., 7)

Construct the bisector  $og$  of the angle  $bod$ .

(Art., 15)

Construct  $ef$  parallel to  $ab$  and at the distance  $r$  from it.

(Art., 20)

Find the intersection  $o'$  of  $ef$  and  $og$ .

(Art., 7)

With  $o'$  as a centre and  $r$  as a radius, describe the given circle.

Proof:  $o'$  is equidistant from  $od$  and  $ob$ .

(W., Art., 162)

It is the distance  $r$  from  $ob$  and therefore also from  $od$ .

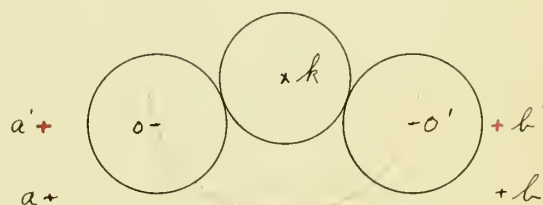
(W., Art., 181)

Hence a circle described with  $o'$  as a centre and  $r$  as a radius will be tangent to  $ab$  and  $cd$  and is the required circle.

II. Touches a given line  $ab$  and a given circle  $k$ .

Let  $ab$  be the given line and  $k$  the given circle.

To construct a circle of radius  $r$  which shall be tangent to  $ab$  and also to  $k$ .



Find the sum of  $r$  and the radius of the circle  $k$ .

(Art., 5)

With this as a radius describe a circle.

Construct  $a'b'$  parallel to  $ab$  and at a distance  $r$  from it.

(Art., 20)

Find the intersections  $o$  and  $o'$  of  $a'b'$  with the circumference.

(Art., 4, II)

With  $o$  and  $o'$  as centres and  $r$  as a radius describe circles.

Then either of these is the required circle.

Proof: By construction,  $o$  and  $o'$  are at the distance  $r$  from  $ab$  and  $r$  plus the radius of  $k$  from  $k$ . Therefore circles described with  $o$  and  $o'$  as centres and  $r$  as a radius will be tangent to  $ab$  and to the circle  $k$ .

III. Passes through a given point  $p$  and touches a given line  $ab$ .

Let  $ab$  be the given line and  $p$  the given point..

To construct a circle of radius  $r$ , tangent to  $ab$  and passing through  $p$ .

Construct  $a'b'$  parallel to  $ab$  and at a distance  $r$  from it.

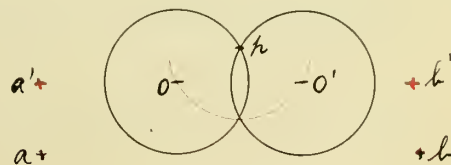
(Art., 20)

With  $p$  as a centre and radius  $r$ , describe a circle and find its intersections  $o$  and  $o'$  with  $a'b'$ .

(Art., 4, II)

With  $o$  and  $o'$  as centres and  $r$  as a radius, describe circles.

Then either of these is the required circle.



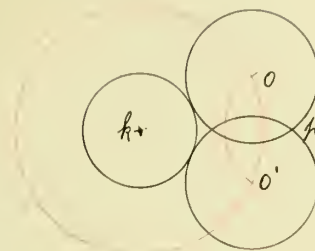


Proof: Both  $o$  and  $o'$  are at the distance  $r$  from  $p$  and  $ab$ . Const. Therefore a circle described with  $o$  or  $o'$  as a centre and  $r$  as a radius will pass through  $p$  and be tangent to  $ab$ .

IV. Passes through a given point  $p$  and touches a given circle  $k$ .

Let  $k$  be the given circle and  $p$  the given point.

To construct a circle of radius  $r$  passing through the point  $p$  and tangent to  $k$ .



Construct  $r$  plus the radius of the circle  $k$ . (Art., 5)

With this radius and  $k$  as a centre describe a circle. With  $p$  as a centre and  $r$  as a radius describe a circle cutting this circumference at  $o$  and  $o'$ . Then with  $o$  or  $o'$  as a centre and  $r$  as a radius describe the required circle.

Proof: By construction, both  $o$  and  $o'$  are at the distance  $r$  from  $p$  and the circumference of  $k$ . Then a circle described with either  $o$  or  $o'$  as a centre and  $r$  as a radius will pass through  $p$  and be tangent to  $k$ .

76. To construct a circle which shall;

I. Touch two given parallels and pass through a given point  $p$ .

Let  $ab$  and  $cd$  be the given parallels and  $p$  the given point.

To construct a circle passing through  $p$  and tangent to both  $ab$  and  $cd$ .

Bisect  $ac$  at  $e$ . (Art., 14)

Construct  $ef$  parallel to  $ab$ . (Art., 20).

With  $p$  as a centre and  $ae$  as a radius, describe a circle, and find its intersections,  $o$  and  $o'$ , with  $ef$ . (Art., 4, II)

With  $o$  or  $o'$  as a centre and  $ae$  as a radius describe the required circle.

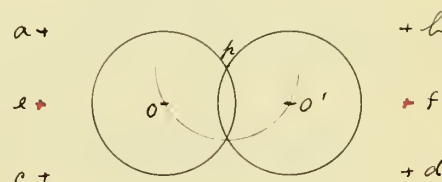
Proof: By construction, either  $o$  or  $o'$  is at the distance  $ae$  from  $p$  and the two parallels  $ab$  and  $cd$ . Then if we take  $o$  or  $o'$  as centre and  $ae$  as radius and describe a circle, it will pass through  $p$  and be tangent to both  $ab$  and  $cd$ .

II. Touch three given lines two of which are parallel.

Let  $ab$  and  $cd$  be the given parallels and  $mn$  the other given line.

To construct a circle tangent to  $ab$ ,  $cd$  and  $mn$ .

Bisect the perpendicular distance  $ac$





at e.

(Art., 14)

Through e construct ef parallel to ab.

(Art., 20)

Construct m'n' parallel to mn and at

(Art., 20)

a distance ae from it.

Find the intersection o of ef and m'n'.

(Art., 7)

With o as a centre and ae as a radius, describe a circle. Then this is the required circle.

Proof: Since o lies in ef it is the distance ae from ab and cd and since it lies in m'n' it is the distance ae from mn. Therefore if we describe a circle with o as a centre and ae as a radius it will be tangent to the three lines ab, cd and mn.

III. Touch a given line ab at p and pass through a given point q.

Let p be some point on the given line ab and q some other given point.

To construct a circle tangent to ab at p and passing through q.

At p, construct pc perpendicular to ab.

(Art., 12, I)

Also construct de perpendicular to pq at its middle point.

(Art., 12, I)

Find the intersection o of pc and de.

(Art., 7)

With o as a centre and op as a radius, describe a circle. Then this is the required circle.

Proof: Since o lies on de, it is equidistant from p and q.

(W., Art., 160)

Therefore the circle passes through q.

Since op is perpendicular to ab the circle is tangent to ab at p.

(W., Art., 253)

Therefore this is the required circle.

IV. Touch a given circle at p and pass through a given point q.

Let p be the given point on the circle o and q any other point.

To construct a circle passing through q and tangent to the circle o at the point p.

Extend the line op to p'.

(Art., 4, Cor)

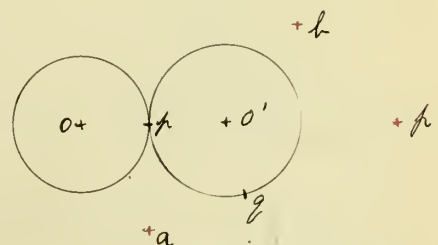
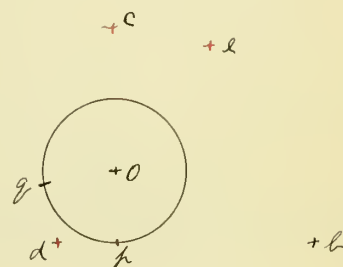
Construct ab perpendicular to pp' at its middle point.

(Art., 12, I)

Find the intersection o' of pp' and ab.

(Art., 7)

With o' as a centre and o'p as a radius, describe a circle.







Then this is the required circle.

Proof: Since  $c'$  lies on  $ab$  it is equidistant from  $p$  and  $q$ .

(W., Art., 160)

Then if we take  $o'$  as a centre and  $o'p$  as a radius and describe a circle it will pass through  $q$ .

V. Touch two given lines and touch one of them at a given point  $p$ .

Let  $ab$  and  $cd$  be the given lines and  $p$  the point on the line  $ab$ .

To construct a circle tangent to  $cd$  and tangent to  $ab$  at the point  $p$ .

Find the intersection  $e$  of  $ab$  and  $cd$ .

(Art., 7)

Construct the bisector  $ef$  of the angle  $bed$ .

(Art., 15)

Erect a perpendicular  $pp'$  to  $ab$  at the point  $p$ .

(Art., 12, I)

Find the intersection  $o$  of  $ef$  and  $pp'$

(Art., 7)

With  $o$  as a centre and  $op$  as a radius, describe a circle. Then this is the required circle.

Proof: The point  $o$  is equidistant from  $ab$  and  $cd$ . (W., Art., 162)

Therefore a circle constructed with  $o$  as a centre and  $op$  as a radius will be tangent to  $ab$  and  $cd$ .

VI. Touch a given line and touch a given circle at a point  $p$ .

Let  $ab$  be the given line and  $p$  the point on the given circle  $o$ .

To construct a circle tangent to  $ab$  and to the given circle at the point  $p$ .

Extend  $op$  to  $p'$ . (Art., 4, Cor)

Construct a perpendicular  $ad$  to  $pp'$  at  $p$  and find its intersection  $c$  with  $ab$ .

(Arts., 12; 7)

Construct the bisector  $ec$  of the angle  $deb$ .

Find the intersection  $o'$  of  $ec$  and  $pp'$ .

(Art., 15)

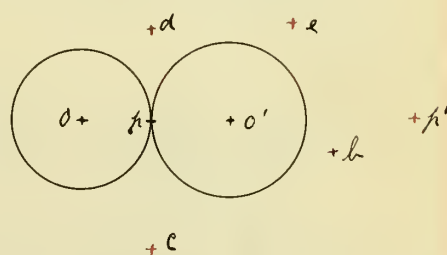
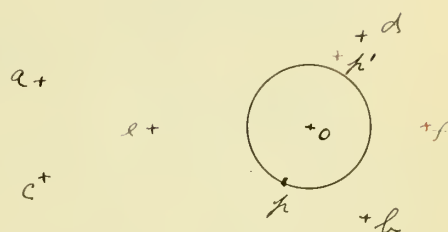
(Art., 7)

With  $o'$  as a centre and  $o'p$  as a radius, describe a circle. Then this is the required circle.

Proof: Since  $o'$  lies on  $ce$  it is equidistant from  $ab$  and  $cd$ .

(W., Art., 162)

Therefore a circle described with  $o'$  as a centre and  $o'p$  as a radius will be tangent to  $ab$  and also to  $cd$  at the point  $p$ . It must, therefore, be tangent to the given circle at the point  $p$ .





VII. Touch a given line  $ab$  at  $p$  and also touch a given circle.

Let  $p$  be the given point on the line  $ab$  and  $o$  the given circle.

To construct a circle tangent to the given circle and tangent to  $ab$  at  $p$ .

Construct  $gd$  perpendicular to  $ab$  at  $p$ .

(Art., 12, I)

Lay off  $pg$  equal to the radius of  $o$ .

(Art., 4, Cor.)

Construct a perpendicular  $ef$  at the middle point  $f$  of  $og$

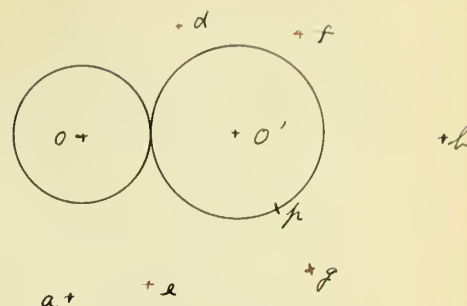
and find its intersection  $o'$  with  $gd$ .

(Arts., 12; 7)

With  $o'$  as a centre and  $o'p$  as a radius, describe a circle. Then this is the required circle.

Proof: The point  $o'$  is equidistant from  $o$  and  $g$ . (W., Art., 160)

Subtracting the radius of  $o$  from each of these distances we get the distance to the circumference of  $o$  equal to  $o'p$ . Therefore a circle described with  $o'$  as a centre and  $o'p$  as a radius will be tangent to  $ab$  and also to the given circle.



77. To inscribe a circle in a given sector.

Let  $ao b$  be the given sector.

To inscribe a circle in  $ao b$ .

Construct the bisector  $oe$  of the angle  $ao b$ .

(Art., 15)

Find the point  $f$  where  $oe$  cuts the arc  $ab$ .

(Art., 4, I)

Construct  $cd$  perpendicular to  $oe$  at the point  $f$  and find its intersections  $c$  and  $d$  with  $ob$  and  $oa$ .

(Arts., 12; 7)

Construct the bisector  $dg$  of the angle  $odc$ .

(Art., 15)

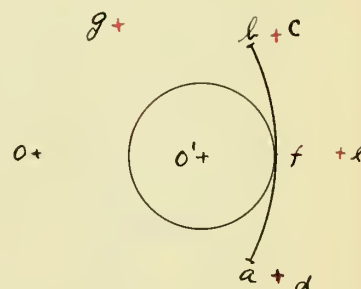
Find the intersection  $o'$  of  $dg$  and  $ce$ .

(Art., 7)

With  $o'$  as a centre and  $o'f$  as a radius, describe a circle. Then this is the required circle.

Proof: The circle  $o'$  is inscribed in the triangle  $odc$ . (Art., 35)

Since it passes through the point  $f$  which is also a point on the arc  $ab$  it is also inscribed in the sector  $ao b$ .



78. To construct within a given circle three equal circles so that each will touch the other and also the given circle.

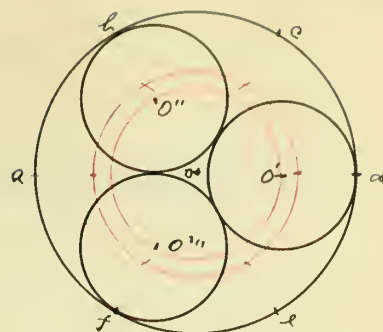
Let  $o$  be the given circle.



To construct three circles within  $o$  which shall be tangent to it and to each other.

Use the radius as a chord and divide the circumference into six equal parts. Inscribe a circle  $o'$  in the sector  $eoc$ .

(Art., 77)



With  $o$  as a centre and  $oo'$  as a radius, describe a circle and find its intersections  $o''$  and  $o'''$  with  $ob$  and  $of$ . (Art., 4, I)  
With  $o''$  and  $o'''$  as centres and  $o'd$  as a radius describe circles. Then  $o', o''$  and  $o'''$  are the required circles.

Proof: Since the two circles  $o'$  and  $o''$  are inscribed in the sectors  $eoc$  and  $coa$  they are both tangent to  $oc$ . Let  $p$  and  $p'$  be their points of tangency. Then in the two right triangles  $oo'p$  and  $oo''p'$ ,  $oo' = oo''$  and  $o'p = o''p'$

Const.

Hence the triangles are equal

(W., Art., 151)

and  $op = op'$ . Therefore the points  $p$  and  $p'$  coincide and the circles are tangent to each other. Similarly for all the other circles. Therefore,  $o', o''$  and  $o'''$  are the required circles.

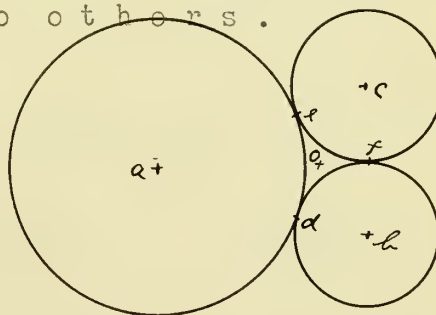
79. To describe circles about the vertices of a given triangle as centres so that each shall touch the two others.

Let  $abc$  be the given triangle.

To describe circles with  $a, b$  and  $c$  as centres, each of which shall touch the other two.

Find the intersection  $o$  of the bisectors of the angles  $a, b$  and  $c$ .

(Art., 35)



Construct perpendiculars  $od, oe$  and  $of$  to the sides of the triangle.

(Art., 12, ID)

Then  $\text{tri. } aoe = \text{tri. } aod$ ,  $\text{tri. } coe = \text{tri. } cof$  and  $\text{tri. } bof = \text{tri. } bod$ . (W., Art., 141)  
Therefore  $ae = ad$ ,  $ce = cf$  and  $bd = bf$ . With  $a, b$ , and  $c$  as centres and radii  $ae, bd$  and  $cf$  respectively, describe three circles. Then these are the required circles.

Proof: Since by construction the distance between the centres of the circles is equal to the sum of their radii, each circle must be tangent to the other two. Therefore these are the required circles.

80. To bisect the angles formed by two





lines, without producing the lines to their point of intersection.

Let  $ab$  and  $cd$  be the given lines.

To bisect the angle between  $ab$  and  $cd$ .

From any point  $c'$  of  $ab$  construct  $c'd'$  parallel to  $cd$ . (Art., 20)

Lay off  $c'e=c'f$ , extend  $ef$  to  $g$  and find its intersection  $g$  with  $cd$ .

(Arts., 4, Cor; 7)

Construct a perpendicular  $mn$  at the middle point of  $eg$ . (Art., 12, D)  
Then  $mn$  is the required bisector.

Proof:  $\angle cge = \angle afe = \angle aeg$

(W., Art., 176)

Therefore  $g, e$  and the point of intersection of  $ab$  and  $cd$  form an isosceles triangle.

(W., Art., 147)

Therefore the perpendicular bisector of  $eg$  bisects the angle between  $ab$  and  $cd$ .

(W., Art., 149)

81. To draw through a given point  $p$  between the sides of an angle  $abc$  a line terminated by the sides of the angle and bisected at  $p$ ;

Let  $abc$  be the given angle and  $p$  the given point.

To draw a line through  $p$  included between  $ab$  and  $bc$  which is bisected at  $p$ .

Construct  $pe$  parallel to  $bc$  and  $pd$  parallel to  $ab$ . (Art., 209)

Find their intersections  $d$  and  $e$  with  $bc$  and  $ab$ .

(Art., 7)

Lay off  $ef=pe$  and  $dg=ep$ .

(Art., 4, Cor.)

Then  $fg$  is the required line.

Proof: Triangle  $fpe =$  triangle  $pdg$ .

(W., Art., 143)

Therefore  $pf=pg$ .

Since the lines  $ep$  and  $bc$  are parallel and  $\angle fpe = \angle hgd$  then  $fpg$  must be a straight line. Therefore it is the required line.

82. Given two points  $p, q$  and a line  $ab$ ; to draw lines from  $p$  and  $q$  which shall meet on  $ab$  and make equal angles with  $ab$ .

Let  $ab$  be the given line and  $p$  and  $q$  the given points.

To draw lines from  $p$  and  $q$  meeting on  $ab$



and making equal angles with  $ab$ .

Construct  $pc$  perpendicular to  $ab$ .

(Art., 12, II)

Find the intersection  $e$  of  $pc$  and  $ab$ .

(Art., 7)

Lay off  $pe = ec$ .

(Art., 4, Cor.)

Find the intersection  $d$  of  $ab$  and  $cq$ .

(Art., 7)

Then  $pd$  and  $qd$  are the required lines.

Proof:  $\angle pda = \angle adc$ ,

(W., Art., 95)

and  $\angle qdb = \angle adc$ .

(W., Art., 93)

Hence  $\angle pda = \angle qdb$ .

Therefore  $pd$  and  $qd$  are the required lines.

83 To find the shortest path from  $p$  to  $q$  which shall touch a line  $ab$ .

Let  $ab$  be the given line and  $p$  and  $q$  the given points.

To find the shortest path from  $p$  to  $q$  which shall touch  $ab$ .

Construct  $pp'$  perpendicular to  $ab$ .

(Art., 12, II)

Find the intersection,  $e$ , of  $ab$  and  $pp'$ .

(Art., 7)

Lay off  $ep' = ep$ .

(Art., 4, Cor.)

Find the intersection,  $c$ , of  $ab$  and  $p'q$ .

(Art., 7)

Then  $pcq$  is the required path.

Proof: Any point in  $ab$  is equidistant from  $p$  and  $p'$ . (W., Art., 160)  
The straight line  $p'cq$  is shorter than any broken line  $p'dq$ . Therefore  $pcq$  is the required shortest path.

84. To draw a common tangent to two given circles.

Let  $M$  and  $M'$  be the given circles.

To draw a common tangent to  $M$  and  $M'$ .

Construct a circle  $M''$  with centre  $o$  and radius equal to the difference of the radii of  $M$  and  $M'$ .

From  $o'$ , construct  $o'p$  and  $o'q$  tangent to  $M''$ .

(Art., 36, II)

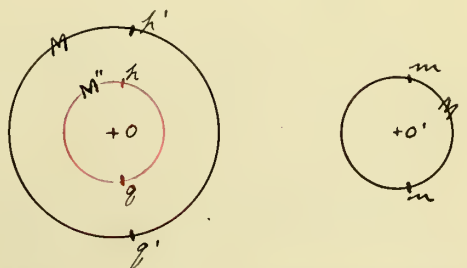
Find the intersections  $p'$  and  $q'$  of  $op$  and  $oq$  with  $M$ .

(Art., 4, I)

Construct  $o'm$  perpendicular to  $o'p$  and  $o'n$  perpendicular to  $o'q$  and find their intersections  $m$  and  $n$  with  $M'$ .

(Arts., 12; 4)

Then  $mp'$  and  $nq'$  are the required tangents.





Proof:  $pp'$  is parallel and equal to  $c'm$ . (W., Art., 104)

Hence  $mp'$  is parallel to  $o'p$  (W., Art., 183)

and must therefore be perpendicular to  $op'$  and  $o'm$ . (W., Art., 107)

Therefore  $mp'$  is tangent to both circles  $M$  and  $M'$ .

Similarly  $nq'$  is a common tangent.

\*Cor. Interior tangents may be constructed by using a circle with radius equal to the sum of the radii of the two given circles.

85. To divide a given straight line into parts proportional to any number of given lines.

Let  $ab$  be the given line and  $m, n$  and  $p$  the given parts.

To divide  $ab$  into parts proportional to  $m, n$  and  $p$ .

On any line  $ac$ , lay off the given parts  $m, n$  and  $p$ . (Art., 4, Cor)  $e'$

Construct  $e'e$  and  $d'd$  parallel to  $b'b$

and find their points of intersection  $d$  and  $e$  with  $ab$ . (Arts., 20; 7)

Then  $ab$  is divided into the required parts by  $d$  and  $e$ .

Proof:  $ad:m=de:n=eb:p$  (W., Art., 344)

Therefore these are the required parts.

86. To find a fourth proportional to three given straight lines.

This problem has been solved. (Art., 6)

87. To find a third proportional to two given straight lines.

Let  $m$  and  $n$  be the given straight lines.

To construct a third proportional to  $m$  and  $n$ .

Take any angle  $asb$  and lay off  $ad=m$  and  $de=ad'=n$ . (Art., 4, Cor.)

Construct  $ee'$  parallel to  $dd'$  (Art., 20)

and find the intersection  $e'$  of  $ee'$  and  $ac$ .

Then  $d'e'$  is the required third proportional.

Proof:  $ad:de=ad':d'e'$ . (W., Art., 344)

Hence  $m:n=n:d'e'$ .

Therefore  $d'e'$  is the required third proportional.

88. Construct  $x$ , if  $x=ab/c$ , (2)  $x=a^2/c$ .





These are special cases of Art., 6.

89. To find the mean proportional between two given straight lines.

Let  $ac$  and  $bc$  be the given straight lines.

To find a mean proportional between  $ac$  and  $bc$ .

Find the sum  $ab$  of  $ac$  and  $bc$ . (Art., 5)

Bisect  $ab$  at  $o$ .

(Art., 14)

Construct a semi-circle with  $oa$  as a radius.

Erect a perpendicular  $cd$  to  $ab$  (Art., 12, I)

and find its intersection  $d$  with the circumference.

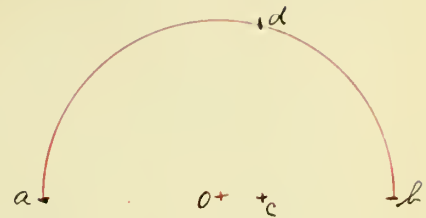
(Art., 4, II)

Then  $cd$  is the mean proportional between  $ac$  and  $bc$ .

Proof:  $ac:cd=cd:ec$ .

(W., Art., 370)

Therefore  $cd$  is the required mean proportional.



90. Construct  $x$ , if  $x=\sqrt{ab}$ .

If in the above construction we make  $a=ac$  and  $b=bc$ , then  $cd=x$  and the construction is completed.

91. To divide a given line in extreme and mean ratio.

Let  $ab$  be the given line.

To divide  $ab$  into extreme and mean ratio.

Construct  $be$  perpendicular to  $ab$  and lay it off equal to  $1/2 ab$ .

(Art., 12; 4, Cor')

With  $e$  as a centre and  $eb$  as a radius describe a circle and find its intersections  $f$  and  $g$  with  $ae$ .

(Art., 4, I)

With  $a$  as a centre and  $af$  and  $ag$  respectively as radii, describe arcs and find their intersections  $c$  and  $c'$  with  $ab$ .

(Art., 4, I)

Then  $ab$  is divided internally at  $c$  and externally at  $c'$  in extreme and mean ratio.

Proof:  $ag:ab=ab:af$

(W., Art., 381)

$$\overline{ab}^2 = af \times ag = ac(ac+ab)$$

$$\overline{ab}^2 = ag \times af = c'a(ag-fg)$$

$$= ac(ac+ab) = \overline{ac}^2 + ab \times ac.$$

$$= c'a(c'a-ab) = \overline{c'a}^2 - ab \times c'a.$$

$$\text{Therefore } \overline{ab}^2 - ab \times ac = \overline{ac}^2.$$

$$\text{Therefore } \overline{ab}^2 + ab \times c'a = \overline{c'a}^2.$$

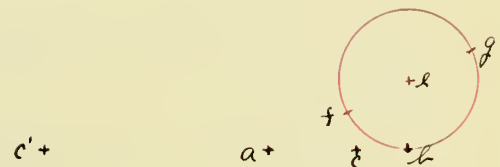
$$\text{Therefore } ab(ab-ac) = \overline{ac}^2.$$

$$\text{Therefore } ab(ab+c'a) = \overline{c'a}^2.$$

$$\text{Therefore } ab \times bc = \overline{ac}^2.$$

$$\text{Therefore } ab \times c'b = \overline{c'a}^2.$$

Q.E.F.





92. Upon a given line homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.

Let  $abcd$  be the given polygon and  $a'b'$  the given line.

To construct a polygon on  $a'b'$  similar to  $abcd$ .

Construct  $\angle c'a'b' = \angle cab$  and  $\angle c'b'a' = \angle cba$ .

(Art., 19)

Find the intersection  $c'$  of  $a'c'$  and  $b'c'$ .

(Art., 7)

Construct  $\angle c'a'd' = \angle cad$  and  $\angle a'c'd' = \angle acd$ .

(Art., 19)

Find the intersection,  $d'$ , of  $c'd'$  and  $a'd'$ .

(Art., 7)

Then  $a'b'c'd'$  is the required polygon.

Proof: Triangle  $abc$  is similar to triangle  $a'b'c'$  and triangle  $acd$  is similar to triangle  $a'c'd'$ .

(W., Art., 355)

Therefore  $abcd$  is similar to  $a'b'c'd'$ .

(W., Art., 365)

93. To divide one side of a given triangle into segments proportional to the adjacent sides.

Let  $abc$  be the given triangle and  $bc$  the given side.

To divide  $bc$  in to segments proportional to  $ab$  and  $ac$ .

Construct the bisector  $ad$  of the angle  $bac$ .

(Art., 15)

Find the intersection  $d$  of  $bc$  and  $ad$ .

(Art., 7)

Then  $bc$  is divided at  $d$  into the required segments.

Proof: (W., Art., 348)

94. To find in one side of a given triangle a point whose distance from the other sides shall be to each other in the given ratio  $m:n$ .

Let  $ab$  be the given side of the triangle  $abc$ , and  $m:n$  the given ratio.

To find a point  $e$  in  $ab$  so that  $ef/ed = m/n$ .

Construct the perpendicular  $ao$  to  $ac$  and lay it off equal to  $m$ . (Arts., 12; 4, Ccr)



Also construct  $oh$  perpendicular to  $bc$  and lay it off equal to  $n$ . Construct  $hg$  parallel to  $bc$  and  $ec$  parallel to  $og$  and find their intersections  $g$  and  $e$  with  $ac$  and  $ab$ . (Art., 20; 7)

Then  $e$  is the required point.

Proof: Construct  $ed$  perpendicular to  $bc$  and  $ef$  perpendicular to  $ac$ . (Art., 12, II)

Find their intersections  $d$  and  $f$  with  $bc$  and  $ac$ . (Art., 7)

Then triangle  $ced$  is similar to triangle  $goh$  and triangle  $cef$  is similar to triangle  $goa$ . (W., Art., 35)

Hence  $ce/ed = go/oh$  and  $ce/ef = go/ao$ . Dividing one of these equations by the other, we get  $ef/ed = ao/oh = m/n$ .

Therefore  $e$  is the required point.

95. Given an obtuse triangle; to draw a line from the vertex of the obtuse angle to the opposite side which shall be the mean proportional between the segments of that side.

Let  $abc$  be the given triangle.

Construct a line  $ec$  so that  $ae:ec = ec:eb$ .

Circumscribe a circle about  $abc$ .

(Art., 34)

Construct a circle on  $oc$  as a diameter.

(Art., 14)

Find its intersections  $e$  and  $e'$  with  $ab$ .

(Art., 4, II)

Then  $ec$  or  $e'c$  is the required line.

Proof: Find the intersection  $d$  of  $ce$  with the circle.  $oe$  is perpendicular to  $dc$ . (Art., 4, II)

(W., Art., 290)

Therefore  $de = ec$

(W., Art., 245)

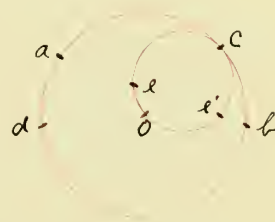
Then  $ae \times eb = de \times ec = ec^2$

(W., Art., 378)

or  $ae:ec = ec:eb$ .

Similarly we get  $ae':e'c = e'c:e'b$ .

Therefore  $ec$  and  $e'c$  are the required lines!



96. Through a given point  $p$  within a given circle to draw a chord  $ab$  so that the ratio  $ap:bp$  shall equal the given ratio  $m:n$ .

Let  $c$  be the given circle,  $p$  the given point and  $m:n$  the given ratio.

To draw a chord  $ab$  through  $p$  so that  $ap:bp$  is equal to  $m:n$ .

Extend  $op$  to  $c$

(Art., 4, Cor.)







making  $op:pc=m:n$

(Art., 6)

Lay off  $ac$  so that  $ac:ob=n:m$ . Extend  $ap$  to  $b$ .

(Art., 4, Cor)

Then  $ab$  is the required chord.

Proof: From the above equations we have  $op:pc=ac:ob$ .

Also  $\angle apc = \angle opb$ .

(W., Art., 93)

Then triangle  $apc$  is similar to triangle  $opb$ .

(W., Art., 357)

Hence  $ap:bp=pc:op=m:n$ .

97. To draw through a given point  $p$  in the arc subtended by a chord  $ab$  a chord which shall be bisected by  $ab$ .

Let  $ab$  be the given chord and  $p$  the given point.

To draw a chord through  $p$  which shall be bisected by  $ab$ .

Find the intersection,  $c$ , of  $op$  and  $ab$ .

(Art., 7)

Lay off  $cd=cp$ .

(Art., 4, Cor)

Construct  $de$  parallel to  $ab$ .

(Art., 20)

Find where it intersects the circumference.

(Art., 4, II)

Find the intersection,  $f$ , of  $ep$  and  $ab$ .

(Art., 7)

Then  $ep$  is the required chord.

Proof: In the triangle  $edp$  the line  $cf$  is constructed from the middle point  $c$  of the side  $dp$ , parallel to  $ed$ . It must, therefore, pass through the middle point  $f$  of  $ep$ .

(W., Art., 188)

Therefore  $ep$  is the required chord.

98. To draw through a given external point  $p$  a secant  $pab$  to a given circle so that the ratio  $pa:ab$  shall equal the given ratio  $m:n$ .

Let  $o$  be the given circle and  $p$  the given external point.

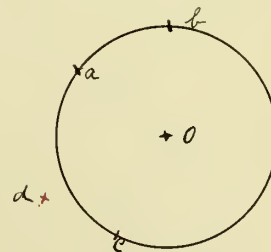
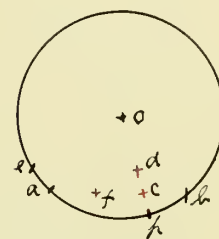
To construct a secant  $pab$  so that  $pa:ab=m:n$ .

Construct the tangent  $pc$  to the circle and divide it at  $d$  in the ratio  $m:n$ .

Construct  $pa$  so that  $pd:pa=pa:pc$ .

Find the intersection,  $b$ , of  $ap$  with the circle.

Then  $pab$  is the required secant.





Proof: We have  $pb:pc=pc:pa$ .

(W., Art., 381)

From this equation and the one above we get

$$pd:pa=pc:pb.$$

Hence triangles  $pda$  and  $pcb$  are similar.

(W., Art., 357)

Therefore  $pa:ab=pd:dc=m:n$ .

Therefore  $pb$  is the required secant.

99. To draw through a given external point  $p$  a secant  $pab$  to a given circle so that  $\overline{ap}^2 = pa \times pb$ .

Let  $o$  be the given circle and  $p$  the given external point.

Construct a secant  $pab$  such that  $ab = pa \times pb$ .

Construct the tangent  $pd$  to the circle and divide it into extreme and mean ratio at  $c$ .

(Arts., 26; 91)

With  $p$  as a centre and  $pa$  as a radius locate the point  $a$  and find the intersection,  $b$ , of  $pa$  with the circle.

(Art., 4, II)

Then  $pab$  is the required secant.

Proof: (1)  $pc:cd=cd:pd$ .

Const.

(2)  $pb:pd=pd:pa=pc:cd$ .

(W., Art., 381)

Combining (1) and (2) we have  $pc:cd$  or  $pc:pa=pd:pb$

Hence the triangles  $pac$  and  $pbd$  are similar.

(W., Art., 357)

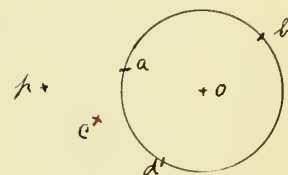
Therefore (3)  $cd:pc=ab:pa$ .

From (1)  $pd=cd^2/pc$  and substituting from (3)  $pd=ab/pa \times pa=ab$

From (2)  $\overline{pd}^2=pb \times pa$

Therefore  $\overline{ap}^2=pb \times pa$ .

Therefore  $pb$  is the required secant.



100. To find a point  $p$  in the arc subtended by a given chord  $ab$  so that the ratio shall equal the given ratio  $m:n$ .

Let  $o$  be the given circle and  $ab$  the given chord.

To find a point  $p$  in the arc  $ab$  such that  $ap:bp=m:n$ .

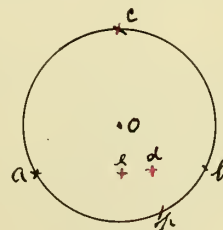
Divide  $ab$  in the ratio  $m:n$  (Art., 85) and let  $d$  be the point of division.

Bisect  $ab$  at  $e$ ,

Erect a perpendicular  $ec$  to  $ab$

and find its intersection  $c$  with the circle.

Find the point  $p$  where  $cd$  cuts the circle.



(Art., 14)

(Arts., 12; 4, I)

(Art., 4, II)



Then  $p$  is the required point.

Proof: Since  $c$  is the middle point of the arc  $acb$  then  $cp$  bisects the angle  $apb$ .

Therefore  $ap:pb=ad:bd=m:n$ .

(W., Art., 348)

Therefore  $p$  is the required point.

101. To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be in the given ratio  $m:n$

Let  $o$  and  $o'$  be the given circles and  $c$  their point of intersection.

To construct a chord  $ab$  such that  $ac:cb=m:n$ .

Divide  $oo'$  in the ratio  $m:n$  (Art., 85)

Let  $d$  be the point of division.

Construct  $acb$  perpendicular to  $cd$  and find their intersection,  $a$  and  $b$ , with the circles.

(Arts., 12; 4. II)

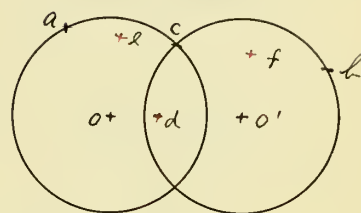
Then  $ab$  is the required chord.

Proof: Construct  $oe$  and  $o'f$  perpendicular to  $ab$ . (Art., 12, II)

Then  $ec:df=ac:cb=od:o'd=m:n$ .

(W., Art., 344)

Therefore  $ab$  is the required chord.



102. Having given the greater segment of line divided in extreme and mean ratio to construct the line.

Let  $m$  be the greater segment.

To construct the line of which  $m$  is the greater segment when it is divided in extreme and mean ratio.

Take any line  $ad$  and construct  $dg$  perpendicular to it. (Art., 12, I)

Lay off  $dg=1/2 ab$ . (Art., 4, Cor.)

With  $a$  as a centre and  $m$  as a radius, describe an arc cutting  $ag$  at  $f$  and  $ad$  at  $e$ .

(Art., 4, I)

Construct  $fe$  perpendicular to  $ag$ .

(Art., 12, I)

Find the intersection  $e$  of  $fe$  and  $ad$ .

(Art., 7)

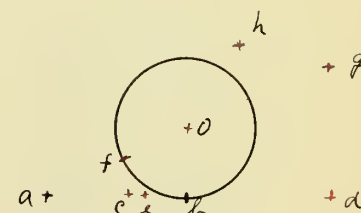
Construct the bisector,  $eh$ , of the angle  $fed$ .

(Art., 15)

Find the intersection  $o$  of  $eh$  and  $ag$ .

(Art., 7)

Construct  $ob$  perpendicular to  $ad$ . Then  $ab$  is the required line.







Proof: The construction is simply the converse of (Art., 91), which is to divide a line in extreme and mean ratio.

103. To construct a circle which shall pass through two given points and touch a given straight line.

Let  $a$  and  $b$  be the given points and  $mn$  the given straight line.

To draw a circle through  $a$  and  $b$  and tangent to  $mn$ .

Find the intersection,  $c$ , of  $ba$  and  $mn$ . (Art., 7)

Construct  $cd$  such that  $ca:cd=cd:cb$ . (Art., 89)

Draw a circle through  $a, b$  and  $d$ . (Art., 34)

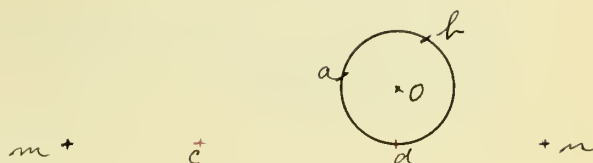
Then this is the required circle.

Proof: It passes through the two points  $a$  and  $b$ . Const.

Since  $ca:cd=cd:cb$

then  $cd$  is tangent to the circle. (W., Art., 381)

Therefore it is the required circle.



104. To construct a circle which shall pass through a given point and touch two given straight lines.

Let  $ab$  and  $ac$  be the two given lines and  $p$  the given point.

To construct a circle passing through  $p$  and tangent to both  $ab$  and  $ac$ .

Construct the bisector  $ad$  of  $\angle bac$ . (Art., 15)

Construct  $pp'$  perpendicular to  $ad$ , (Art., 12, II)

Find the intersection  $e$  of  $pp'$  and  $ad$ . (Art., 7)

Lay off  $ep' = ep$ . (Art., 4, Cor.)

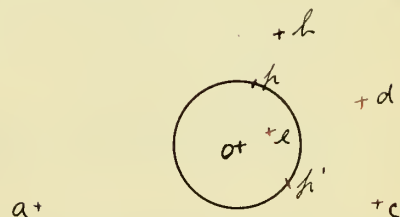
Pass a circle through  $p$  and  $p'$  and tangent to  $ac$ . (Art., 103)

Then this is the required circle.

Proof: Since the circle passes through  $p$  and  $p'$ , its centre must lie on  $ad$ . (W., Art., 248)

Since the centre lies on  $ad$  and the circle touches  $ac$ , it must also touch  $ab$ . (W., Art., 162)

Therefore  $o$  is the required circle.

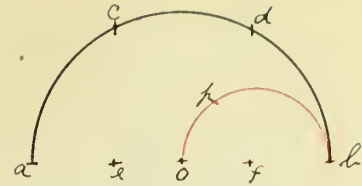


105. To inscribe a square in a semi-circle. Let  $o$  be the centre of the given semi-circle,  $acdb$ .



To inscribe a square in acdb.

Construct a semi-circle with ob  
as a diameter. (Art., 14)  
Locate p such that op:pb=1:2. (Art., 100)



Lay off oe=of=op and fd=ec=bp.  
Then ecdf is the required square.

(Art., 4, Cor.)

Proof: Triangle ofd=triangle oec=triangle opb. (W., Art., 150)  
Then  $\angle oec = \angle ofd = \angle opd =$  a right angle. (W., Art., 290)  
Also of+oe=ef=fd=ec=cd.  
Therefore ecdf is the required square.

106. To inscribe a square in a given triangle.

Let fbe be the given triangle.

To inscribe a square in fbe.

Let a'b'c'd' be the required square.

Construct bc parallel to ef, (Art., 20)

Produce fc'

and find the intersection, c, of fc' and bc.

Construct ba and cd perpendicular to fe.

Find the intersections, a and d, with ef.

Then tri.fb'c' is similar to tri.fbc and tri.fc'd' is similar to tri.fcd.

Hence  $fc':fc=c'd':cd=c'b':cb$ .

Since  $c'b'=c'd'$ , then  $cd=cb$  and since d' is a right angle, then d is a right angle and therefore abcd is a square.

Therefore to find the point c, construct the square abcd and find the intersection of fc and be.

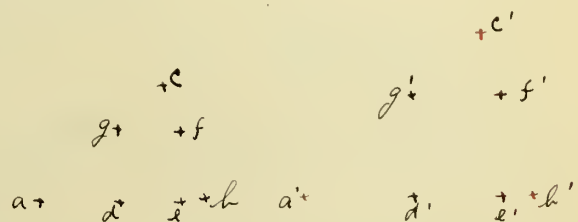
Having found the point c' we can easily construct the required square a'b'c'd' by constructing perpendicular and parallel lines.

107. To inscribe in a given triangle a rectangle similar to a given rectangle.

Let abc be the given triangle and d'e'f'g' the given rectangle.

To inscribe in abc a rectangle similar to d'e'f'g'.

Place the triangle abc so that ab will be parallel to d'e'. Through g' and f' construct c'a' parallel to ca





and  $c'b'$  parallel to  $cb$ .

(Art., 20)

Extend  $d'e'$  to  $a'$  and  $b'$ .

(Art., 4, Cor.)

Then triangles  $a'b'c'$  and  $abc$  are similar.

(W., Art., 354)

Locate  $g$  so that  $ag:gc=a'g':g'c'$ .

(Art., 85)

Construct  $gf$  parallel to  $ab$  and then complete the rectangle  $defg$ . Then this is the required rectangle.

Proof:  $\text{tri. } c'gf$  is similar to  $\text{tri. } c'g'f'$  and  $\text{tri. } agd$  to  $\text{tri. } a'g'd'$ .

(W., Art., 354)

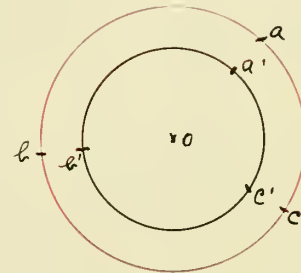
Hence  $fg:f'g'=gd:g'd'$ .

(W., Art., 351)

Therefore  $defg$  is the required rectangle.

108. To inscribe in a circle a triangle similar to a given triangle.

Let the smaller circle whose centre is  $o$  be the given circle and  $abc$  the given triangle.



To inscribe in  $c$  a triangle similar to  $abc$ .

Circumscribe a circle around  $abc$ .

(Art., 34)

Describe the given circle concentric to this circle.

Find the intersections  $a', b'$  and  $c'$  of  $oa, ob$  and  $oc$  with this circle.

(Art., 4, I)

Then  $a'b'c'$  is the required triangle.

Proof: Triangles  $abc$  and  $a'b'c'$  are similar. (W., Art., 354)

Therefore  $a'b'c'$  is the required triangle.

109. To inscribe in a given semicircle a rectangle similar to a given rectangle.

Let  $o'$  be the given semicircle and  $defg$  the given rectangle.

To inscribe in  $o'$  a rectangle similar to  $defg$ .

With  $o$ , the middle point of  $de$ , as a centre and of  $oa$  as a radius, describe a semicircle.

Construct  $o'e'$  such that  $o'e':o'f'=pe:of$ .

(Art., 6)

Lay off  $o'd'=o'e'$ .

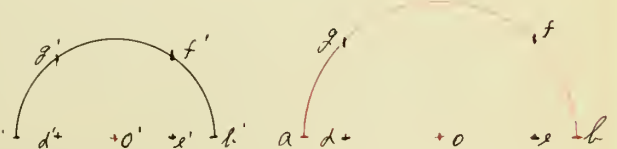
(Art., 4, Cor)

Construct  $d'g'$  and  $e'f'$  perpendicular to  $a'b'$

and find their intersections  $g'$  and  $f'$  with the semicircle. (Arts., 12, 4)

Then  $d'e'f'g'$  is the required rectangle.

Proof: The triangles  $o'e'f'$  and  $oef$  are similar. (W., Art., 354)







Then  $o'e':oe=e'd':ed=e'f':ef$  etc.

(W., Art., 358)

Therefore  $d'e'f'g'$  is the required rectangle.

110. To circumscribe about a circle a triangle similar to a given triangle.  $a'$   
+

Let  $M$  be the given circle and  $abc$  the given triangle.

To circumscribe a triangle about  $M$  similar to  $abc$ .

Circumscribe a circle  $M'$  about the given triangle  $abc$ . (Art., 34)

Construct the given circle  $M$  concentric to  $M'$ .

Construct  $ae$  perpendicular to  $ac$ ,  $of$  to  $ab$  and  $od$  to  $bc$ . (Art., 12, II)

Find their intersections  $e, f$  and  $d$  with the circle  $M$ . (Art., 4, I)

Construct tangents to  $M$  at  $e, f$  and  $d$

and find their intersections  $a', b'$  and  $c'$ .

(Arts., 36; 7)

Then  $a'b'c'$  is the required triangle.

Proof: Triangles  $abc$  and  $a'b'c'$  are similar.

(W., Art., 354)

Therefore  $a'b'c'$  is the required triangle.

111. To construct the expression  $x = 2abc/de$ ; that is  $2ab/d \times c/e$ .

This simply involves multiplication and division of lines.

See (Arts., 8, 9)

112. To construct two straight lines, having given their sum and their ratio.

Let  $ab$  be their sum and  $m:n$  their ratio.

To divide  $ab$  in the ratio  $m:n$ , see

(Art., 85)

$$\begin{array}{ccccccc} a & + & & & + & b \\ + & m & + & & & \\ + & n & + & & & \end{array}$$

113. To construct two straight lines, having given their difference and their ratio.

Let  $a-b$  be their difference and  $m:n$  their ratio.

To construct  $a$  and  $b$ .

Construct  $m-n$ . (Art., 5)

Construct a fourth proportional to  $(m-n)$ ,  $n$  and  $(a-b)$ . (Art., 6)

This fourth proportional is  $b$ , for

$$\begin{array}{ccccccc} + & a-b & + & & & \\ + & m & + & & & \\ + & n & + & & & \\ + & a-b & + & b & + & \\ + & m-n & + & & & \\ + & n & + & & & \end{array}$$



$$(m-n):n=(a-b):b.$$

(W., Art., 383)

Then construct  $(a-b)+b=a$ .

(Art., 5)

Therefore we have constructed  $a$  and  $b$ .

114. Given two circles, with centres  $o$  and  $o'$ , and a point  $A$  in their plane, to draw through the point  $A$  a straight line, meeting the circumference at  $A$  and  $B$  so that  $AB:AC=m:n$ .

Let  $M$  and  $M'$  be the given circles.,A the given point and  $m:n$  the given ratio.To construct a line through  $A$  meeting  $M$  and  $M'$  at  $B$  and  $C$  so that  $AB:AC=m:n$ .Extend  $oA$  to  $D$  making  $m:n=oA:AD$ 

(Art., 4, Cor; 6)

Construct  $DC$  such that  $m:n=oB:oC$ . (Art., 6)With  $D$  as a centre and radius  $DC$ , locate the point  $C$ .Find the intersection,  $B$ , of  $CA$  and  $M$ .

(Art., 4, II)

Then  $BC$  is the required line.Proof: The triangles  $AoB$  and  $DAC$  are similar. (W., Art., 345)Then  $AB:AC=oA:AD=m:n$ .Therefore  $BC$  is the required line.

115. To construct a square equivalent to the sum of two given squares.

+ +

A

B

a

C

Let  $A$  and  $B$  be the two given squares.To construct a square equivalent to  $A+B$ .Construct a right angle  $abc$ . (12, I)Lay off  $ab$  equal to a side of  $A$  and  $bc$  equal to a side of  $B$ . (Art., 4, Cor.)Construct a square,  $C$ , with  $ac$  as a side.

(Arts., 12; 4)

Then  $C$  is the required square.Proof:  $\overline{ac}^2 = \overline{ab}^2 + \overline{bc}^2$ 

(W., Art., 415)

Therefore  $C$  is equivalent to  $A+B$ .

\*Cor. By taking  $ac$  and the side of a third given square, we would be able to construct a square equivalent to the sum of three squares. This can be continued indefinitely.



116. To construct a square equivalent to the difference of two given squares.



Let A and B be the two given squares.

To construct a square equivalent to A-B.

Construct a right angle abd.

(Art., 12)

Lay off ab equal to a side of B,

(Art., 4, Cor.)

With a as a centre and a radius equal to the side of A, strike an arc cutting bd at c.

(Art., 4, II)

Construct a square, C, with side bc.

(Arts., 12; 4)

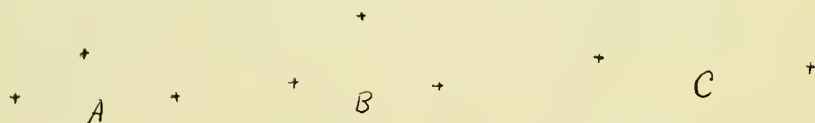
Then C is the required square.

Proof:  $bc^2 = ac^2 - ab^2$

(W., Art., 416)

Therefore C is equivalent to A-B.

117. To construct a polygon similar to two given similar polygons and equivalent to their sum.



Let A and B be the two similar polygons and ab and a'b' the two homologous sides.

To construct a similar polygon equivalent to A+B.

Construct a''b'' so that  $a''b''^2 = a'b'^2 + ab^2$ . (Art., 115)

On a''b'' construct a polygon C similar to B. (Art., 92)

Then C is the required polygon.

Proof:  $a''b''^2 = a'b'^2 + ab^2$

Const.

Also  $A:C = ab^2 : a''b''^2$  and  $B:C = a'b'^2 : a''b''^2$

(W., Art., 412)

By addition  $A+B:C = (ab^2 + a'b'^2) : a''b''^2 = 1$ .

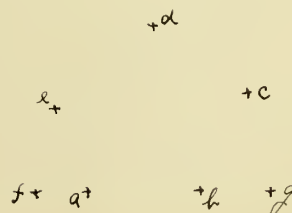
Therefore C is equivalent to A+B.

118. To construct a triangle equivalent to a given polygon.

Let abcde be the given polygon.

To construct a triangle equivalent to abcde.

Construct cg parallel to bd







and  $ef$  parallel to  $ad$ ,

(Art., 20)

and find their intersections  $g$  and  $f$  with  $ab$ .

(Art., 7)

Then  $dfg$  is the required triangle.

Proof:  $\text{tri.dbg}$  is equivalent to  $\text{tri.dbc}$  and  $\text{tri.def}$  is equivalent to  $\text{tri.dae}$ .

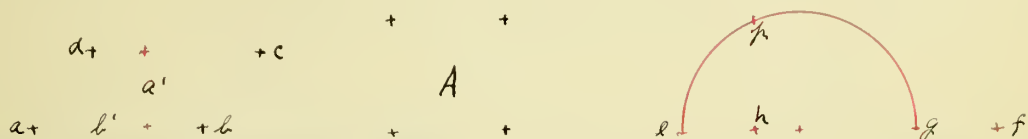
(W., Art., 404)

In the polygon  $abcde$  and the  $\text{tri.dfg}$ , the part  $adb$  is common,  $\text{tri.dbg}$  is equivalent to  $\text{tri.dbc}$  and  $\text{tri.daf}$  to  $\text{tri.dae}$ .

Therefore  $\text{tri.fdg}$  is equivalent to  $abcde$  and is the required triangle.

\*Cor. This same process may be applied to a polygon of any number of sides.

119. To construct a square equivalent to a given parallelogram.



Let  $abcd$  be the given parallelogram,  $b'$  its base and  $a'$  its altitude. To construct a square equivalent to the parallelogram  $abcd$ .

Construct a mean proportional  $hp$  between  $a'$  and  $b'$ . (Art., 89)

Construct a square  $A$  having  $hp$  as a side.

(Art., 12, 4)

Then  $A$  is the required square.

Proof:  $hp = eh \times gh = a' \times b'$ .

(W., Art., 370)

Therefore  $A$  is equivalent to the parallelogram  $abcd$ .

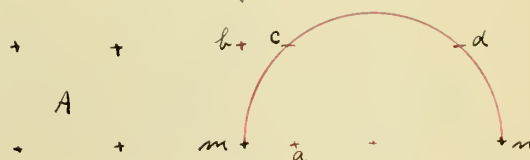
\*Cor. I. A square may be constructed equivalent to a given triangle, by taking for its side the mean proportional between the base and half the altitude of the triangle.

\*Cor. II. A square may be constructed equivalent to a given polygon, by first reducing the polygon to an equivalent triangle, and then constructing a square equivalent to the triangle.

120. To construct a parallelogram equivalent to a given square and having the sum of its base and altitude equal to a given line.

Let  $A$  be the given square and  $mn$  the sum of the base and altitude of the required parallelogram.

To construct a parallelogram equivalent to  $A$  and having the sum of its base and altitude equal to  $mn$ .





Construct a semicircle upon  $mn$  as a diameter. (Art., 14)

At  $m$ , erect a perpendicular  $mb$  to  $mn$

and lay it off equal to the side of the given square  $A$ . (Art., 12, 4)

Construct  $bd$  parallel to  $mn$

and find its intersections  $c$  and  $d$  with the circle. (Arts., 20, 4)

From  $c$  construct  $ca$  perpendicular to  $mn$ . (Art., 12, II)

Then any parallelogram constructed with  $an$  as a base and  $am$  as an altitude will satisfy the given conditions.

Proof:  $\overline{ac}^2 = \overline{bm}^2 = A$ .

Also  $\overline{ac}^2 = am \times an$ .

(W., Art., 370)

Therefore  $A$  is equivalent to  $am \times an$ .

121. To construct a parallelogram equivalent to a given square and having the difference of its base and altitude equal to a given line.

Let  $A$  be the given square and  $mn$  the difference between the base and altitude of the required parallelogram.

To construct a parallelogram equivalent to  $A$  and having the difference of its base and altitude equal to  $mn$ .

Construct a circle upon  $mn$  as a diameter. (Art., 14)

At  $m$  erect a perpendicular  $ma$  to  $mn$

and make it equal to the side of the given square,  $A$ . (Arts., 12, 4)

Find the intersections,  $b$  and  $c$ , of  $ao$  with the circle. (Art., 4, I)

Then a parallelogram constructed having  $ac$  as a base and  $ab$  as an altitude will satisfy the given conditions.

Proof:  $\overline{am}^2 = A$ .

Const.

$\overline{am}^2 = ac \times ab$ .

(W., Art., 381)

$ac - ab = bc = mn$ .

Therefore  $A$  is equivalent to  $ac \times ab$ .

122. To construct a polygon similar to a given polygon  $P$  and equivalent to a given polygon  $Q$ .

Let  $P$  and  $Q$  be the two given polygons, and  $ab$  a side of  $P$ .

To construct a polygon similar to  $P$  and equivalent





to Q.

Find squares equivalent to P and Q  
and let m and n respectively denote their sides. (Art., 119, Cor. II)  
Construct  $m:n=ab:a'b'$ . (Art., 6)  
Upon  $a'b'$ , homologous to  $ab$ , construct  $P'$  similar to P. (Art., 92)  
Then  $P'$  is equivalent to Q and is the required polygon.

Proof:  $m:n=ab:a'b'$ . Const.  
Hence  $m^2:n^2=ab^2:a'b'^2$ . (W., Art., 338)  
But  $P \approx m^2$  and  $Q \approx n^2$ . Const.  
Hence  $P:Q=m^2:n^2=ab^2:a'b'^2$ .  
But  $P:P'=ab^2:a'b'^2$ . (W., Art., 412)  
Then  $P:Q=P:P'$ .  
Therefore  $P' \approx Q$ .

123. To construct a polygon similar to two given similar polygons and equivalent to their difference.

Let P and Q be the two given similar polygons.

To construct a polygon R similar to P and Q and equivalent to their difference.

Construct  $a''b''$  so that  $a'b'^2 - ab^2 = a''b''^2$ . (Art., 116)  
On  $a''b''$  construct a polygon similar to P. (Art., 92)  
Then this is the required polygon.

Proof: The proof is the same as for (Art., 117) after changing  $A+B$  to  $A-B$ .

124. To construct a square which shall have a given ratio to a given square.

Let A be the given square and  $n:m$  the given ratio.

To construct a square which shall be to A as  $n:m$ .

Construct a fourth proportional  $b$  to  $m, n$  and  $a$ . (Art., 6)  
Construct a mean proportional  $x$  to  $a$  and  $b$ . (Art., 89)  
Erect perpendiculars at the extremities of  $x$  and lay them off equal to  $x$ . (Arts., 12; 4, Cor.)  
Then this is the required square.







Proof:  $a:x=x:b$ .

(W., Art., 270)

Multiplying both sides of the equation by  $a/x$  we get  $a^2:x^2=a:b$

But  $a:b=m:n$ .

Therefore  $a^2:x^2=m:n$  or  $x^2:a^2=n:m$ .

Hence the square described on  $x$  will have the same ratio to  $A$  as  $n:m$ .

125. To construct a polygon similar to a given polygon and having a given ratio to it.

Let  $A$  be the given polygon and  $n:m$  the given ratio.

To construct a polygon similar to  $A$ , which shall be to  $A$  as  $n:m$ .

Construct  $a'b'$  such that  $a'b'^2:ab^2=n:m$ .

(Art., 124)

Upon  $a'b'$ , as a side homologous to  $ab$ , construct the polygon  $B$  similar to  $A$ .

(Art., 92)

Then  $B$  is the required polygon.

Proof:  $B:A=a'b'^2:ab^2$ .

(W., Art., 412)

But  $a'b'^2:ab^2=n:m$ .

Const.

Therefore  $B:A=n:m$ .

126. To construct a triangle equivalent to a given triangle and having one side equal to a given length  $l$ .

Let  $abc$  be the given triangle and  $a'b'$  equal to the given side  $l$ .

To construct a triangle equivalent to  $abc$  and having  $a'b'$  for one side.

Construct  $cd$  perpendicular to  $ab$ .

(Art., 12, II)

Find the intersection,  $d$ , of  $ab$  and  $cd$ .

(Art., 7)

Construct  $a'd'$  such that  $a'b':ab=cd:a'd'$ .

(Art., 6)

Then any triangle having  $a'b'$  for a base and  $a'd'$  for an altitude will satisfy the given conditions.

Proof:  $a'b':ab=cd:a'd'$ .

Const.

Hence  $a'b' \times a'd'$  = twice the area of triangle  $a'b'c'$

and  $ab \times cd$  = twice the area of triangle  $abc$ .

(W., Art., 403)

Therefore  $abc$  is equivalent to  $a'b'c'$ .

129. To transform a given triangle into



an equivalent right triangle.

Let  $abc$  be the given triangle.

To construct a right triangle  
equivalent to  $abc$ .

Construct  $cd$  perpendicular to  $ab$ ,

(Art., 12, II)

Find its intersection,  $d$ , with  $ab$ .

(Art., 7)

Construct a right triangle  $a'b'c'$ , having  $ab$  and  $cd$  as legs. (Arts., 12;  
4, Cor)

Then  $a'b'c'$  is the required triangle.

Proof:  $ab \times cd = a'b' \times c'a'$ .

Const.

But  $ab \times cd =$  twice the area of the triangle  $abc$ ,

and  $a'b' \times c'a' =$  twice the area of the triangle  $a'b'c'$ . (W., Art., 403)

Therefore  $\text{tri. } abc$  is equivalent to  $\text{tri. } a'b'c'$ .

128. To transform a given triangle into  
an equivalent right triangle, having one  
leg equal to a given length.

Let  $abc$  be the given triangle and

$a'b'$  the given leg.

To transform  $abc$  into an equivalent  
right triangle having  $a'b'$  as one leg.

Construct  $a'c'$  such that  $a'b':ab=cd:a'c'$ . (Art., 6)

Then a right triangle constructed having  $a'b'$  and  $a'c'$  as legs will be  
the required triangle. (Arts., 12, 4, Cor)

Proof:  $a'b' \times a'c' = ab \times cd$ .

Const.

But  $a'b' \times a'c' =$  twice the area of the triangle  $a'b'c'$

and  $ab \times cd =$  twice the area of the triangle  $abc$ . (W., Art., 403)

Therefore  $\text{tri. } abc$  is equivalent to  $\text{tri. } a'b'c'$ .

129 To transform a given triangle into  
an equivalent right triangle, having the  
hypotenuse equal to a given length.

Let  $abc$  be the given triangle

and  $a'b'$  the given hypotenuse.

To construct on the hypotenuse  
 $a'b'$  a right triangle equivalent to  
the triangle  $abc$ .

Construct  $cd$  perpendicular to  $ab$  and  $a'd'$  to  $a'b'$ . (Art., 12)

Construct a semicircle with  $a'b'$  as a diameter. (Art., 14)



Construct  $a'd'$  such that  $a'b':ab=cd:a'd'$ . (Art., 6)  
 Construct  $d'c'$  parallel to  $a'b'$ .  
 and intersecting the semicircle at  $c'$  and  $c''$ . (Arts., 20; 4, II)  
 Then  $a'b'c'$  or  $a'b'c''$  is the required triangle.

Proof:  $\angle a'c'b' = \angle a'c''b' =$  a right angle. (W., Art., 290)  
 $ab \times cd = a'b' \times a'd'$ . Const.

But  $ab \times cd =$  twice the area of the triangle  $abc$   
 and  $a'b' \times a'd' =$  twice the area of the triangle  $a'b'c'$ . (W., Art., 403)  
 Therefore  $\text{tri. } abc$  is equivalent to  $\text{tri. } a'b'c'$ .

Note: The problem is impossible if  $a'd'$  greater than  $1/2 a'b'$ .

130. To transform a triangle  $abc$  into an equivalent triangle, having a side equal to a given length  $l$ , and an angle equal to an angle  $bac$ .

Let  $abc$  be the given triangle,  $ad=l$   
 the given side and  $bac$  the given angle.

To construct a triangle equivalent to  $abc$ , having  $ad$  as one side and  $bac$  as an angle.

Construct  $be$  parallel to  $cd$  and find its intersection,  $e$ , with  $ac$ .

Then  $aed$  is the required triangle.

Proof:  $\text{tri. } bec \cong \text{tri. } bed$ . (W., Art., 404)  
 Therefore  $\text{tri. } aed \cong \text{tri. } abc$ .

131. To transform a given triangle into an equivalent isosceles triangle, having the base equal to a given length.

Let  $abc$  be the given triangle  
 and  $a'b'$  the given base.

To construct an isosceles triangle having  $a'b'$  as a base and equivalent to the triangle  $abc$ .

Construct  $c'd'$  perpendicular to  $a'b'$  at its middle point. (Art., 12, I)  
 Construct  $c'd'$  such that  $a'b':ab=cd:c'd'$ . (Art., 6)  
 Then  $a'b'c'$  is the required triangle.

Proof:  $c'a' = c'b'$ . (W., Art., 953)  
 Hence triangle  $a'b'c'$  is isosceles.

Also  $ab \times cd =$  twice the area of the triangle  $abc$   
 and  $a'b' \times c'd' =$  twice the area of the triangle  $a'b'c'$ . (W., Art., 403)





Therefore  $\text{tri.abc}$  is equivalent to  $\text{tri.a'b'c'}$ .

132. To construct a triangle equivalent to;

I. The sum of two given triangles.

Let  $\text{abc}$  and  $\text{a'b'c'}$  be the two given triangles.

To construct a triangle equivalent to the sum of  $\text{abc}$  and  $\text{a'b'c'}$ .

$$\begin{array}{ccccccc} & & & & +c' & & +c'' \\ +c & & & & & & \\ a+ & +b & a' & +b' & a'' & +b'' & \end{array}$$

Construct squares A and B equivalent to the triangles  $\text{abc}$  and  $\text{a'b'c'}$ .

(Art., 119, Cor., I)

Construct a square  $\text{C}$  equivalent to  $A+B$ .

(Art., 115)

Construct a triangle  $\text{a''b''c''}$  equivalent to C.

(Art., 118)

Then  $\text{a''b''c''}$  is the required triangle.

II. The difference of two given triangles.

Let  $\text{abc}$  and  $\text{a'b'c'}$  be the two given triangles.

$$\begin{array}{ccccccc} & & & & +c' & & +c'' \\ & & & & & & \\ a''+ & +b'' & a'+ & +b' & a+ & +b & \end{array}$$

To construct a triangle equivalent to the difference of  $\text{abc}$  and  $\text{a'b'c'}$ .

Construct squares A and B equivalent to the triangles  $\text{abc}$  and  $\text{a'b'c'}$ .

Construct a square C equivalent to  $A-B$ .

(Art., 116)

Construct a triangle  $\text{a''b''c''}$  equivalent to C.

(Art., 118)

Then  $\text{a''b''c''}$  is the required triangle.

133. To transform a given triangle into an equivalent equilateral triangle.

Let  $\text{abc}$  be the given triangle.

To construct an equilateral triangle equivalent to  $\text{abc}$ .

Construct  $\text{cd}$  perpendicular to  $\text{ab}$ .

(Art., 12, II)

Find the intersection,  $\text{d}$ , of  $\text{ab}$  and  $\text{cd}$ .

(Art., 7)

$$\begin{array}{ccccccc} & & & & +c' & & \\ & & & & & & \\ a+ & +a & +b & +b' & +d' & +b' & \end{array}$$

Construct  $2 \times \overline{\text{ab} \times \text{cd}}$ ,

(Art., 8)

and then  $\sqrt{2 \times \overline{\text{ab} \times \text{cd}}}$ .

(Art., 10)

Construct  $\sqrt[3]{3} = \sqrt[3]{3}$ ,

(Art., 10)

and then  $(\sqrt{2 \times \overline{\text{ab} \times \text{cd}}}) \div \sqrt[3]{3}$ .

(Art., 9)

Construct the equilateral triangle  $\text{a'b'c'}$ , making  $\text{a'b'} = (\sqrt{2 \times \overline{\text{ab} \times \text{cd}}}) \div \sqrt[3]{3}$ .

(Art., 32)



Then  $a'b'c'$  is the required triangle.

Proof:  $c'd'^2 = (\sqrt{2 \times ab \times cd}) + \sqrt[4]{3}^2 - 1/4 (\sqrt{2 \times ab \times cd}) + \sqrt[4]{3}^2 = 1/2 ab \times cd \sqrt{3}$ . (W., Art. 371)

Hence  $c'd' = (\sqrt{ab \times cd \times \sqrt[4]{3}}) + \sqrt{2}$ .

Triangle  $a'b'c' \cong 1/2 (\sqrt{ab \times cd \times \sqrt[4]{3}}) + \sqrt{2} (\sqrt{2 \times ab \times cd}) + \sqrt[4]{3} = 1/2 ab \times cd$ . (W., Art., 403)

But triangle  $abc = 1/2 ab \times cd$ , (W., Art., 403)

and therefore triangle  $abc \cong$  triangle  $a'b'c'$ .

Therefore  $a'b'c'$  is the required triangle.

134. To transform a parallelogram into an equivalent:

I. Parallelogram having one side equal to a given length.

Let  $abcd$  be the given parallelogram and  $a'b'$  the given side.

To construct a parallelogram with a side  $a'b'$  and equivalent to  $abcd$ .

Construct  $ef$  perpendicular to  $ab$  and  $e'f'$  to  $a'b'$ . (Art., 12)

Construct  $e'f'$  such that  $a'b':ab = ef:e'f'$ . (Art., 6)

Construct  $d'c'$  parallel to  $a'b'$  and at a distance  $e'f'$  from it. (Art., 20)

With  $a'$  and  $b'$  as centres and any radius describe arcs cutting  $b'c'$  in  $b''$  and  $c'$ . (Art., 4)

Then  $a'b'c'd'$  is the required parallelogram.

Proof:  $ab \times ef = a'b' \times e'f'$ . Const.

But  $ab \times ef =$  area of the parallelogram  $abcd$ .

and  $a'b' \times e'f' =$  area of the parallelogram  $a'b'c'd'$ . (W., Art., 400)

Therefore, parallelogram  $abcd \cong$  parallelogram  $a'b'c'd'$ .

II. Parallelogram having one angle equal to a given angle.

Let  $abcd$  be the given parallelogram and  $b'a'g'$  the given angle.

To construct a parallelogram

equivalent to  $abcd$  and having  $b'a'g'$

as one angle.

Lay off  $a'b'$  any arbitrary length.

(Art., 4, Cor)

Construct  $ef$  perpendicular to  $ab$  and  $e'f'$  to  $a'b'$ . (Art., 12)

Lay off  $e'f'$  such that  $a'b':ab = ef:e'f'$ . (Art., 6)

Construct  $d'e'c'$  parallel and equal to  $a'b'$ . (Art., 20)

Then  $a'b'c'd'$  is the required parallelogram.

Proof:  $\angle d'a'b'$  is the given angle.



$$ab \times ef = a'b' \times e'f'.$$

Const.

But  $ab \times ef$  = area of the parallelogram  $abcd$

and  $a'b' \times e'f'$  = area of the parallelogram  $a'b'c'd'$ . (W., Art., 400)

Therefore, parallelogram  $abcd \cong$  parallelogram  $a'b'c'd'$ .

### III. Rectangle having a given altitude.

Let  $abcd$  be the given parallelogram and  $a'b'$  the altitude of the required rectangle.

To construct a rectangle with altitude  $a'd'$  and equivalent to  $abcd$ .

$$\begin{array}{ccccccc} a_+ & +e & +c & a'_+ & +c' \\ a_+ & +f & +b & a'_+ & +b' \end{array}$$

Construct  $a'b'$  such that  $a'd':ef=ab:a'b'$ .

(Art., 6)

Erect perpendiculars  $a'd'$  and  $b'c'$  to  $a'b'$

and lay them off equal to  $a'd'$ .

(Arts., 12; 4, Cor.)

Then  $a'b'c'd'$  is the required rectangle.

Proof:  $ab \times ef = a'b' \times a'd'$ .

Const.

But  $ab \times ef$  = area of the parallelogram  $abcd$ ,

(W., Art., 400)

and  $a'b' \times a'd'$  = area of the rectangle  $a'b'c'd'$ .

(W., Art., 298)

Therefore, parallelogram  $abcd \cong$  rectangle  $a'b'c'd'$ .

### 135. To transform a square into an equivalent.

#### I. Equilateral triangle.

Let  $abcd$  be the given square.

To construct an equilateral triangle equivalent to  $abcd$ .

$$\begin{array}{ccc} a_+ & +c & +c' \end{array}$$

Construct triangle  $ade \cong$  square  $abcd$ .

$$\begin{array}{ccccccc} a_+ & +b & +e & a'_+ & +b' \end{array}$$

(Art., 118)

Construct the equilateral triangle  $a'b'c' \cong$  triangle  $ade$ . (Art., 133)

Then  $a'b'c'$  is the required triangle.

Proof: Triangle  $ade \cong$  square  $abcd$ .

Const.

Triangle  $ade \cong$  triangle  $a'b'c'$ .

Const.

Therefore, triangle  $a'b'c' \cong$  square  $abcd$ .

#### II. Right triangle having one leg equal to a given length.

Let  $abcd$  be the given square and  $a'b'$  the given leg of the required triangle.

$$\begin{array}{ccc} a_+ & +c & +c' \\ & & +d' \end{array}$$

To construct a right triangle equivalent to  $abcd$  and having  $a'b'$  for one leg.

$$\begin{array}{ccccccc} a_+ & +b & a'_+ & +b' \end{array}$$





Construct  $b'd'$  such that  $a'b':ab=ab:b'd'$ . (Art., 6)

Construct  $b'c'$  perpendicular to  $a'b'$  and equal to  $2b'd'$ . (Art., 12; 8)

Then  $a'b'c'$  is the required triangle.

Proof:  $\overline{ab}^2 = a'b' \times b'd'$ . Const.

But  $\overline{ab}^2 = \text{area of the square } abcd$ . (W., Art., 398)

And  $1/2 a'b' \times c'b' = a'b' \times b'd' = \text{area of triangle } a'b'c'$ . (W., Art., 403)

Therefore triangle  $a'b'c' \approx \text{square } abcd$ .

III. Rectangle having one side equal to a given length.

Let  $abcd$  be the given square and  $a'b'$  the side of the required rectangle.

To construct a rectangle equivalent to  $abcd$  and having  $a'b'$  as one side.

Construct  $c'b'$  such that  $a'b':ab=ab:c'b'$ . (Art., 87)

Construct  $a'd'$  and  $b'c'$  perpendicular to  $a'b'$  and lay them off equal to  $c'b'$ .

(Arts., 12; 4, Cor.)

Then  $a'b'c'd'$  is the required rectangle.

Proof:  $\overline{ab}^2 = a'b' \times c'b'$  Const.

But  $\overline{ab}^2 = \text{area of the square } abcd$ . (W., Art., 398)

And  $a'b' \times c'b' = \text{area of rectangle } a'b'c'd'$  (W., Art., 398)

Therefore, square  $abcd \approx \text{rectangle } a'b'c'd'$ .

136. To construct a square equivalent to;

I. Five eighths of a given square.

Let  $abcd$  be the given square.

To construct a square equivalent to five eighths of  $abcd$ .

Construct a line equal to  $5/8 ab$ . (Arts., 8, 9)

Construct  $a'b'$  such that  $ab:a'b'=a'b':5/8 ab$ . (Art., 89)

Construct  $a'd'$  and  $b'c'$  perpendicular to  $a'b'$  and lay them off equal to  $a'b'$ .

(Arts., 12; 4, Cor.)

Then  $a'b'c'd'$  is the required square.

Proof:  $\overline{a'b'}^2 = 5/8 \overline{ab}^2$ . Const.

Therefore area of square  $a'b'c'd' \approx 5/8 \text{ area of square } abcd$ .

II. Three fifths of a given pentagon.

Let  $abcde$  be the given pentagon.

To construct a square equivalent to three fifths of  $abcde$ .

Construct triangle  $A \approx \text{pentagon}$



abcde.

(Art., 118)

Construct square  $E \approx$  triangle A.

(Art., 119, Cor. I)

Construct square  $a'b'c'd' \approx 3/5$  square B.

(Art., 126, I)

Then  $a'b'c'd'$  is the required square.

Proof: Pentagon abcde  $\approx$  triangle A  $\approx$  square B,

Const.

and square  $a'b'c'd' \approx 3/5$  square B.

Const.

Therefore square  $a'b'c'd' \approx 3/5$  pentagon abcde.

137. To divide a given triangle into two equivalent parts by a line through a given point p in one of the sides.

Let abc be the given triangle and p the given point in the side ac.

$a^+ \quad +c$   $d' \quad +c'$

To draw a line through p dividing abc into two equivalent parts.

$a^+ \quad +b$   $a' \quad +b'$

Construct  $pp'$  perpendicular to ab.

(Art., 12, II)

Construct square  $a'b'c'd' \approx$  triangle abc.

(Art., 119, Cor. I)

Construct ad such that  $pp':a'b':a'b':ad$ .

(Art., 87)

Lay off ad on ab.

(Art., 4, Cor.)

Then pd divides the triangle into the required parts.

Proof:  $pp' \times ad = a'b'^2$ .

Const.

Hence  $pp' \times ad =$  area of triangle abc.

But  $1/2 pp' \times ad =$  area of triangle adp.

(Art., 403)

Therefore triangle adp  $\approx 1/2$  triangle abc.

Then  $pdbc \approx 1/2$  triangle abc.

Note. If ad is greater than ab, use the perpendicular from p to bc instead of  $pp'$  and lay off ad on cb.

138. To find a point within a triangle, such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.

Let abc be the given triangle.

$+c \quad +g'$

To find a point p such that triangle apb  $\approx$  triangle apc  $\approx$  triangle bpc.

Construct cd perpendicular to ab and be to ac.

$+a \quad +b$   $+g$

(Art., 12, II)

Construct  $1/3$  cd and  $1/3$  be. (Art., 23)

$a^+ \quad +d$   $+b$

Construct fg parallel to ab and at a distance  $1/3$  cd from it and  $f'g'$  parallel to ac and at a distance  $1/3$  be from it.

(Art., 20)

Find the intersection, p, of fg and  $f'g'$ .

(Art., 7)



Then p is the required point.

Proof: Area of triangle apc =  $\frac{1}{2} ac \times ep = \frac{1}{2} ac \times \frac{1}{3} eb$ . (W., Art., 4C3)  
 But  $\frac{1}{2} ac \times \frac{1}{3} eb = \frac{1}{3}$  triangle abc.  
 Similarly triangle apb =  $\frac{1}{3}$  triangle abc.  
 Therefore p is the required point.

139. To divide a given triangle into two equivalent parts by a line parallel to one of the sides.

Let abc be the given triangle.

To divide abc into two equivalent parts by a line parallel to ab.

Construct cd perpendicular to ab and find its intersection, d, with ab.

Construct  $cd' = cd/\sqrt{2}$ .

Lay off  $cd'$  on cd.

Construct a'b' through d' parallel to ab and find its intersections with ac and bc.

Then a'b' is the required line.

Proof: Triangle abc and a'b'c' are similar.

Therefore  $\text{tri.abc}:\text{tri.a'b'c'} = cd^2:cd'^2 = 2:1$

Hence  $\text{tri.a'b'c'} \approx \frac{1}{2} \text{tri.abc}$ .

Then  $\text{abb'a'} \approx \frac{1}{2} \text{tri.abc}$ .

Therefore a'b' is the required line.

+c

a' + d' + b'

a + d + b

(Arts., 12; 7)

(Arts., 9; 10)

(Art., 4, Cor.)

(Arts., 20-7)

(Art., 359)

(W., Art., 412)

140. To divide a given triangle into two equivalent parts by a line perpendicular to one of the sides.

Let abc be the given triangle.

To draw a line perpendicular to ab dividing abc into two equivalent parts.

Construct cd and ef perpendicular to ab and suppose ef is the required line.

For brevity let  $be=y$ ,  $ef=x$ ,  $bd=m$ ,  $ad=n$ ,  $cd=p$ .

Then  $\frac{1}{2} p(n+m) = \text{area of triangle abc}$ .

And (1)  $\frac{1}{2} xy = \frac{1}{4} p(n+m)$ .

Also  $y:x=m:p$ .

Therefore  $mx=py$  or  $x=py/m$ .

Substituting this in (1) and solving, we get

$y^2 = (np+p^2) \div 2$  or  $y = \sqrt{(np+p^2) \div 2}$  or  $be = \sqrt{(ad \times cd + cd^2) \div 2}$ .

Hence, construct  $be = \sqrt{(ad \times cd + cd^2) \div 2}$

+c

+f

b + e + d + a

(Art., 12, II)

(W., Art., 403)

Const.

(W., Art., 342)

(Arts., 7, 8, 9, 10)





From e construct ef perpendicular to ab.

(Art., 12, I)

Then ef is the required line.

Proof: From equation (1),  $\frac{1}{2} ef \times be = \frac{1}{4} cd \times ab$ .

Therefore triangle bef  $\approx$   $\frac{1}{2}$  triangle abc.

Therefore ef is the required line.

141. To inscribe a square in a given circle.

Let o be the given circle.

To inscribe a square in the circle c.

Construct two diameters, ac and bd, perpendicular to each other. (Art., 12)

Find their intersections, a, b, c and d with the circle. (Art., 4, I)

Then abcd is the required square.

Proof: Angles abc, bcd, etc., are right angles.

(W., Art., 290)

Also  $ab = bc = cd = da$ .

(W., Art., 241)

Hence abcd is a square.

\*Cor. By bisecting the arcs ab, bc, etc., and repeating the operation, we can form regular inscribed polygons of eight, sixteen, etc., sides.

142. To inscribe a regular hexagon in a given circle.

Let o be the given circle.

To inscribe a regular hexagon in o.

Begin with any point, as a, on the circumference and apply the radius, oa, six times as a chord, cutting the circumference at b, c, d, etc.

Then abcdef is the inscribed hexagon.

Proof:  $ab = bc = cd = \text{etc.}$

The triangle aob is equiangular.

Const.

(W., Art., 146)

Hence  $\angle aob = 60^\circ$  and arc  $ab = 60^\circ = \frac{1}{6}$  of a circumference.

Therefore ab is the side of a regular inscribed hexagon.

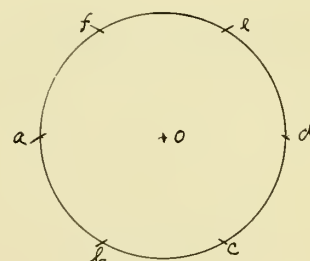
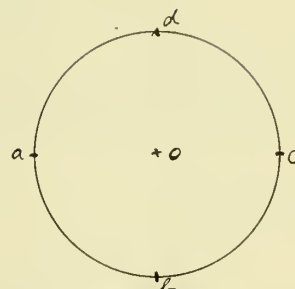
\*Cor. I. ace is an inscribed equilateral triangle.

\*Cor. II. By bisecting the arcs ab, bc, etc., and continuing the process we get regular inscribed polygons of twelve, twenty-four, etc., sides.

143. To inscribe a regular decagon in a given circle.

Let c be the given circle.

To inscribe a regular decagon inc.





Divide the radius  $oa$  into extreme and mean ratio such that  $oa:oc=oc:ac$ .

(Art., 91)

Beginning with  $a$  as a centre and using  $oc$  as a radius, describe arcs locating the points  $b, d$ , etc.

Then  $abd$ .... is the required decagon.

Proof:  $oa:oc=oc:ac$  and  $oc=ab$ .

Const-

Hence  $oa:ab=ab:oc$ .

Also  $\angle a$  is common to the two triangles  $oab$  and  $cab$ .

Hence the triangles  $oab$  and  $cab$  are similar.

(W., Art., 357)

Since tri.  $oab$  is isosceles, then tri.  $abc$  is isosceles and  $ab=bc=oc$ .

Then tri.  $bco$  is isosceles and  $\angle o = \angle cbo$ .

But, exterior  $\angle acb = \angle o + \angle cbo = 2\angle c$ .

(W., Art., 137)

Then  $\angle oab = \angle oba = 2\angle o$ .

Hence the sum of the angles of the tri.  $abo = 5\angle o = 2$  right angles.

Then  $\angle o = 1/5$  of 2 right angles  $= 1/10$  of 4 right angles.

Therefore, arc  $ab$  is  $1/10$  of the circumference and chord  $ab$  is the side of a regular inscribed decagon.

\*Cor. I. By joining the alternate vertices of a regular inscribed decagon, a regular pentagon is formed.

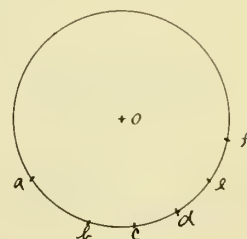
\*Cor. II. By bisecting the arcs  $ab, bd$ , etc., and continuing the process we get regular inscribed polygons of twenty, forty, etc., sides.

144. To inscribe in a given circle a regular pentadecagon, or polygon of fifteen sides.

Let  $o$  be the given circle.

To inscribe in  $o$  a regular polygon of fifteen sides.

Lay off the chord  $ac$  equal to the radius of the circle and  $ab$  equal to the side of a regular inscribed decagon.



Then  $bc$  is the side of the required polygon and applying it fifteen times as a chord we get the required polygon.

Proof: The arc  $ac$  is  $1/6$  of the circumference. (Art., 142)

The arc  $ab$  is  $1/10$  of the circumference.

Const.

Hence, the arc  $bc$  is  $1/6 - 1/10$  or  $1/15$  of the circumference.

Then  $bc$  is the side of a regular inscribed pentadecagon.

\*Cor. By bisecting the arcs  $bc, cd$ , etc., we get regular polygons of thirty, sixty, etc., sides.

145. To inscribe in a given circle a reg-



ular polygon similar to a given regular polygon.

Let  $o'$  be the given circle and  $abcdef$  the given regular polygon.

To inscribe in  $o'$  a regular polygon similar to  $abcdef$ .

Let  $o$  be the centre of the given polygon.

At  $o'$  construct  $\angle e'o'd' = \angle eod$ .

(Art., 19)

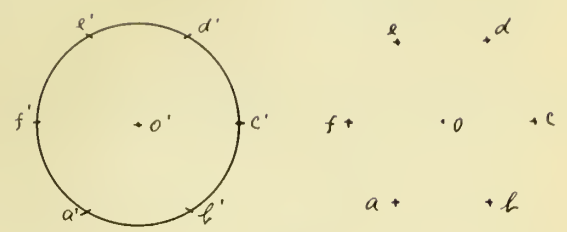
Find the intersections,  $e'$  and  $d'$ , of  $oe'$  and  $od'$  with the circle.

(Art., 4, I)

Then  $e'd'$  is the side of the required polygon and applying it six times as a chord we get the required polygon.

Proof: Each polygon has as many sides as the  $\angle c$ , or  $\angle o'$ , is contained times in four right angles. Therefore, the polygon  $a'b'c'$  is similar to the polygon  $abc$ ....etc.

(W., Art., 445)



146. To circumscribe an equilateral triangle about a given circle.

Let  $o$  be the given circle.

To circumscribe an equilateral triangle about  $c$ .

Divide the given circumference into three equal parts by  $a, b$  and  $c$ .

(Art., 142, Cor. I)

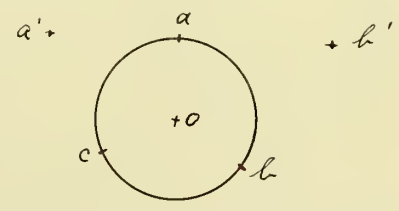
Construct tangents to the circle at  $a, b$  and  $c$ ,

and find their intersections  $a'b'c'$ .

Then  $a'b'c'$  is the required triangle.

Ercof:  $abc$  is a regular inscribed triangle. Const.

Therefore  $a'b'c'$  is a regular circumscribed triangle. (W., Art., 449)



(Arts., 36, 7)

147. To circumscribe a square about a given circle.

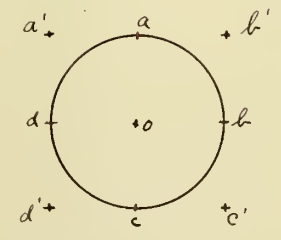
Let  $o$  be the given circle.

To circumscribe a square about  $o$ .

Inscribe a square,  $abcd$ , in the given circle.

(Art., 141)

Construct tangents to the circle at the points  $a, b, c, d$ , and find their intersections  $a'b'c'd'$ . (Arts., 36; 7)







Then  $a'b'c'd'$  is the required square.

Proof:  $abcd$  is an inscribed square.

Then  $a'b'c'd'$  is a circumscribed square.

Const.

(W., Art., 44C)

148. To circumscribe a regular hexagon about a given circle.

Let  $o$  be the given circle.

To circumscribe a regular hexagon about  $o$ .

Inscribe a regular hexagon in  $o$ . (Art., 142)

Construct tangents to  $o$  at the points  $a, b, c$  etc.

and find their intersections  $a', b', c'$  etc. (Arts., 36; 7)

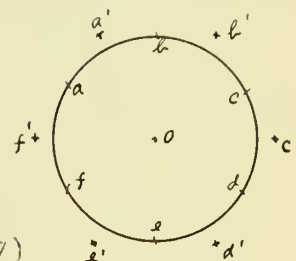
Then  $a'b'c'd'e'f'$  is the required hexagon.

Proof:  $abcdef$  is a regular inscribed hexagon.

Const.

Then  $a'b'c'd'e'f'$  is a regular circumscribed hexagon.

(W., Art., 44C)



149. To circumscribe a regular octagon about a given circle.

Let  $o$  be the given circle.

To circumscribe a regular octagon about  $o$ .

Inscribe a regular octagon in  $o$ .

(Art., 141, Cor.)

Construct tangents to the circle at the vertices  $a, b, c$ , etc., of the regular inscribed octagon, and find their intersections  $a', b', c'$ , etc.

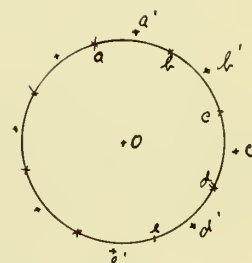
Then  $a'b'c'$ , etc., is the required octagon.

Proof:  $abcd$  is a regular inscribed octagon.

Const.

Then  $a'b'c'd'$ .... is a regular circumscribed octagon.

(W., Art., 44C)



(Arts., 36; 7)

150. To circumscribe a regular pentagon about a given circle.

Let  $o$  be the given circle.

To circumscribe a regular pentagon about  $o$ .

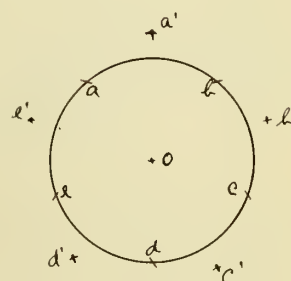
Inscribe a regular pentagon in  $o$ .

(Art., 143, Cor. I)

Construct tangents to the circle at the vertices  $a, b, c$ , etc., of the regular inscribed pentagon,

and find the points of intersection,  $a', b', c'$  etc., of these tangents.

Arts., 36, 7)





Then a'b'c'd'e' is the required pentagon.

Proof: abcde is a regular inscribed pentagon. Const.

Then a'b'c'd'e' is a regular circumscribed pentagon. (W., Art., 44C)

151. To draw through a given point a line so as to divide a given circumference into two parts having the ratio 3:7.

Let o be the given circle and p the given point.

To draw through p a line dividing the circumference of o into two parts having the ratio 3:7.

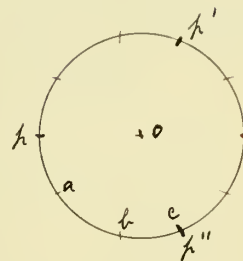
Inscribe in o a regular decagon using p as one of the vertices.

Then pp' or pp'' cutting off three of the equal arcs is the required line.

Proof: Since it is a regular polygon the subtended arcs are equal.

(W., Art., 241)

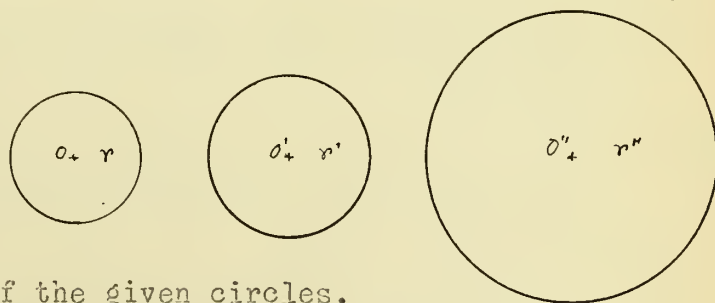
Then a line cutting off three of them would divide the circumference in the ratio 3:7.



152. To construct a circumference equal to the sum of two given circumferences.

Let o and o' be the two given circles.

To construct a circle whose circumference will be the sum of the circumferences of o and o'.



Let r and r' be the radii of the given circles.

Construct a circle o'' with  $r'' = r + r'$

(Art., 5)

Then o'' is the required circle.

Proof: The circumference of the circles are respectively  $2\pi r$ ,  $2\pi r'$  and  $2\pi r''$ .

(W., Art., 458)

We must have, therefore,  $2\pi r'' = 2\pi r + 2\pi r'$  or  $r'' = r + r'$ . Hence if we construct a circle with radius  $r'' = r + r'$  it will be the required circle.

153. To construct a circumference equal to the difference of two given circumferences.

Let o' and o'' be the two given circles.

To construct a circle whose circumference will be the difference of



the circumferences of  $o''$  and  $o'$ .

Let  $r'$  and  $r''$  be the radii of the given circles.

Construct a circle  $o$  with  $r=r''-r'$ .

(Art., 5)

Then  $o$  is the required circle.

Proof: The circumference of the circles are respectively  $2\pi r, 2\pi r', 2\pi r''$ .

(W., Art., 458)

We must, therefore, have  $2\pi r = 2\pi r'' - 2\pi r'$  or  $r = r'' - r'$ . Hence if we construct a circle with radius  $r = r'' - r'$  it will be the required circle.

154. To construct a circle equivalent to the sum of two given circles.

Let  $o$  and  $o'$  be the two given circles.

To construct a circle equivalent to the sum of the two given circles.

Let  $r$  and  $r'$  be the radii of the given circles.

Construct  $r'' = \sqrt{r^2 + r'^2}$ .

(Art., 13, Cor.)

Then construct a circle  $o''$  with  $r''$  as a radius. Then  $o''$  is the required circle.

Proof: Squaring  $r'' = \sqrt{r^2 + r'^2}$  and multiplying by  $\pi$  we get

$$\pi r''^2 = \pi r^2 + \pi r'^2. \text{ Hence } o'' \approx o + o'. \text{ (W., Art., 463)}$$

155. To construct a circle equivalent to the difference of two given circles.

Let  $o'$  and  $o''$  be the two given circles.

To construct a circle equivalent to the difference of the two given circles.

Let  $r'$  and  $r''$  be the radii of the given circles.

Construct  $r = \sqrt{r''^2 - r'^2}$

(Art., 13, Cor.)

Then construct a circle  $o$  with the radius  $r$ . Then  $o$  is the required circle.

Proof: squaring  $r = \sqrt{r''^2 - r'^2}$  and multiplying by  $\pi$  we get

$$\pi r^2 = \pi r''^2 - \pi r'^2. \text{ Hence } o \approx o'' - o'. \text{ (W., Art., 463)}$$

156. To construct a circle equivalent to three times a given circle.

Let  $o$  be the given circle.





To construct a circle equivalent to three times  $o$ .

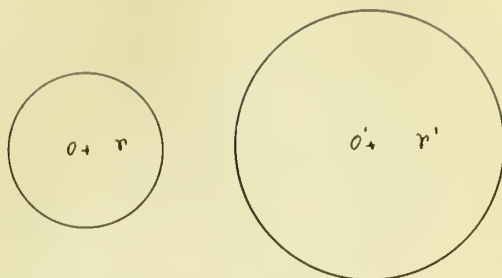
Let  $r$  be the radius of the given circle.

Construct  $r' = r\sqrt{3}$ . (Arts., 10; 8)

Construct a circle  $o'$  with radius  $r'$ .

Then  $o'$  is the required circle.

Proof: Squaring  $r' = r\sqrt{3}$  and multiplying by  $\pi$  we get  $\pi r'^2 = 3\pi r^2$ .  
Hence  $o' \approx 3 \times o$ . (W., Art., 463)



157. To construct a circle equivalent to three-fourths of a given circle.

Let  $o$  be the given circle.

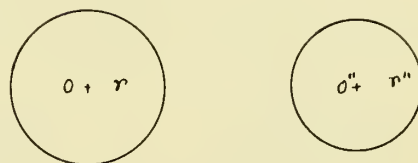
To construct a circle equivalent to three-fourths of  $o$ .

Let  $r$  be the radius of the given circle.

Take  $r'$  of (Art., 156) and construct  $r'' = 1/2r' = 1/2r\sqrt{3}$ . (Art., 14)

Then a circle,  $o''$ , described with  $r''$  as a radius will be the required circle.

Proof: Squaring  $r'' = 1/2r\sqrt{3}$  and multiplying by  $\pi$  we get  
 $\pi r''^2 = 3\pi r^2 / 4$ . Hence  $o'' \approx 3/4 o$ . (W., Art., 463)



158. To construct a circle whose ratio to a given circle shall be equal to the given ratio  $m:n$ .

Let  $o$  be the given circle and  $m:n$  the given ratio.

To construct a circle  $o'$  such that  $m:n = o:o'$ .

Let  $r$  be the radius of the given circle and construct  $r^2$ . (Art., 8, Cor.)

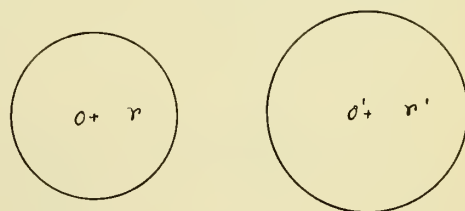
Construct  $r'^2$  such that  $m:n = r^2:r'^2$ .

From  $r'^2$  construct  $r'$ .

Then construct a circle with radius  $r'$ . Then this is the required circle.

Proof:  $m:n = r^2:r'^2$ .

Hence  $m:n = o:o'$ .



(Art., 6)

(Art., 10)

Const.

(W., Art., 464)

159. To divide a given circle by a concentric circumference into two equivalent parts.

Let  $r$  be the radius of the given circle.

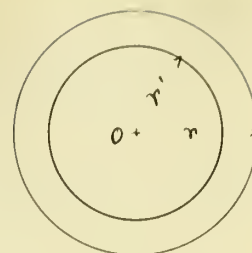


To construct a circumference with  $o$  as a centre which shall divide the given circle into two equivalent parts.

Construct  $\sqrt{2}$ . (Art., 10)

Then construct  $r'$  such that  $r' = r/\sqrt{2}$ . (Art., 9)

Then a circle constructed with  $r'$  as a radius will be the required circle.



Proof: Squaring  $r' = r/\sqrt{2}$  and multiplying by  $\pi$  we get  $\pi r'^2 = \pi r^2/2$ .

Hence the circle with radius  $r' \approx 1/2$  the circle with radius  $r$ . (W., Art., 462)

Therefore the circle with radius  $r$  is divided into two equivalent parts by the circumference of  $r'$ .

160. To divide a given circle by concentric circumferences into five equivalent parts.

Let  $o$  be the centre of the given circle, and  $r$  its radius.

To divide  $o$  by concentric circumferences into five equivalent parts.

Construct  $\sqrt{5}$ . (Art., 10)

Also construct  $r/\sqrt{5}$  (Art., 9)

Then  $\sqrt{2} \times r/\sqrt{5}$ ,  $\sqrt{3} \times r/\sqrt{5}$ ,  $\sqrt{4} \times r/\sqrt{5}$ .

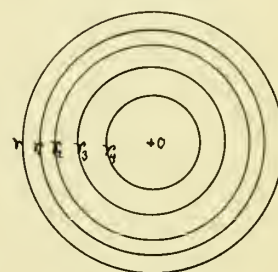
(Art., 8)

Construct concentric circumferences with radii  $r_4 = r/\sqrt{5}$ ,  $r_3 = \sqrt{2} \times r/\sqrt{5}$ ,  $r_2 = \sqrt{3} \times r/\sqrt{5}$ ,  $r_1 = 2 \times r/\sqrt{5}$ . Then these are the required circumferences.

Proof: The areas of the respective circles are  $\pi r^2$ ,  $\pi r^2/5$ ,  $2\pi r^2/5$ ,  $3\pi r^2/5$ ,  $4\pi r^2/5$ .

(W., Art., 463)

Hence these are the required circumferences.



161. To construct an angle of  $36^\circ$ .

Let  $o$  be any circle and construct  $ab$  as the side of a regular inscribed decagon. (Art., 143)

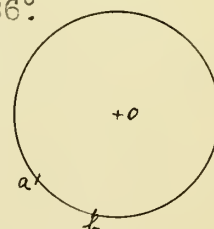
Then  $\angle aob$  is the required angle.

Proof:  $\angle aob = 1/10$  of  $360^\circ = 36^\circ$ .

(W., Arts., 243; 237)

\*Cor. Bisecting  $\angle aob$  we get an angle of  $18^\circ$  and bisecting this again we get an angle of  $9^\circ$ .

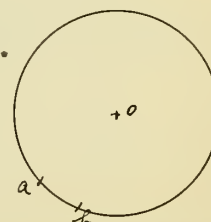
(Art., 15)



162. To construct an angle of  $24^\circ$ .

Let  $o$  be any circle and construct  $ab$  as the side of a regular inscribed pentadecagon. (Art., 144)

Then  $\angle aob$  is the required angle.





Proof:  $\angle aob = 1/15$  of  $360^\circ = 24^\circ$ .

(W., Arts., 248, 237)

\*Cor. Bisecting  $\angle aob$  we get an angle of  $12^\circ$  and bisecting this again we get an angle of  $6^\circ$ . (Art. 15)

163. To construct with a side of a given length;

I. An equilateral triangle.

+c

Let  $ab$  be the given side.

To construct an equilateral triangle having  $ab$  as a side.

With  $a$  and  $b$  as centres and  $ab$  as a radius describe arcs intersecting at  $c$ . Then  $abc$  is the required triangle.

a+ +b

II. A square.

d+ +c

Let  $ab$  be the given side.

To construct a square having a side  $ab$ .

At  $a$  and  $b$  construct  $bc$  and  $ad$  perpendicular to  $ab$ . (Art., 12, I)

a+ +b

Lay off  $ad=bc=ab$ . (Art., 4, Cor.)

Then  $abcd$  is the required square.

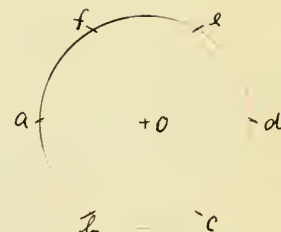
III. A regular hexagon.

Let  $oa$  be the given side.

To construct a regular hexagon with  $oa$  as a side.

Draw a circle,  $o$ , with  $oa$  as a radius and within it inscribe a regular hexagon. (Art., 142)

Then  $abcdef$  is the required hexagon.



Proof:  $ab=bc=cd=de=ef=fa$ . Const.

Also  $ab=oa$ .

Const.

Therefore  $abcdef$  is the required hexagon.

IV. A regular octagon.

Let  $a'b'$  be the given side.

a'+ +b'

To construct a regular octagon having  $a'b'$  as a side.

Construct  $\angle mbn = 135^\circ$ .

(Art., 16)

Lay off  $ab=bc=a'b'$ .

(Art., 4, Cor.)

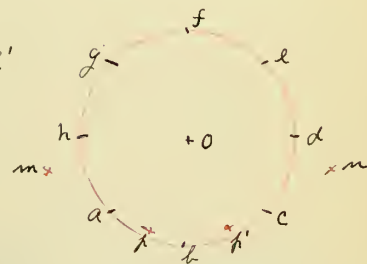
Construct perpendiculars,  $po$  and  $p'o$ , at the middle points of  $ab$  and  $bc$  and find their intersection  $o$ . (Arts., 12, 7)

Construct a circle with  $o$  as a centre and  $oa$  as a radius and lay off the chords  $ab=bc=cd=de$ .

Then  $abcde...$  is the required octagon.

Proof:  $\angle abc = 135^\circ$ .

Const.







Hence  $\angle aob = 45^\circ$ .

(W., Art., 438)

Therefore acbde... is a regular octagon.

V. A regular pentagon.

Let a'b' be the given side.

To construct a regular pentagon having a'b' as a side.

Construct  $\angle nbm = 108^\circ$ . (Art., 143, Cor, I)

Lay off  $ab = bc = a'b'$ . (Art., 4, Cor.)

Construct perpendiculars, op and op', at the middle points of ab and bc

and find their intersection o. (Arts., 12, 7)

Construct a circle with o as a centre and ao as a radius and lay off chords  $ab = bc = cd = de$ . Then abcde is the required pentagon.

Proof:  $\angle abc = 108^\circ$ .

Const.

Hence  $\angle aob = 72^\circ$ .

(W., Art., 438)

Therefore abcde is a regular pentagon.

VI. A regular decagon.

Let a'b' be the given side.

To construct a regular decagon having a'b' as a side.

Construct  $\angle mbn = 144^\circ$ . (Art., 143)

Lay off  $ab = bc = a'b'$ . (Art., 4, Cor.)

Construct perpendiculars, op and op', at the middle points of ab and bc and find their intersection o. (Arts., 12, 7)

Construct a circle with o as a centre and oa as a radius and lay off chords  $ab = bc = cd = etc.$  Then abcd... is the required decagon.

Proof:  $\angle abc = 144^\circ$ .

Const.

Hence  $\angle aob = 36^\circ$ .

(W., Art., 438)

Therefore abcd... is the required decagon.

VII. A regular dodecagon.

Let a'b' be the given side.

To construct a regular dodecagon having a'b' as a side.

Construct  $\angle mbn = 150^\circ$ . (Art., 17, Cor)

Lay off  $ab = bc = a'b'$ . (Art., 4, Cor.)

Construct perpendiculars, po and p'o, at the middle points of ab and bc and find their intersection o. (Art., 12, 7)

Construct a circle with o as a centre and oa as a radius and lay off the chords  $ab = bc = cd = etc.$  Then abcd... is the required dodecagon.

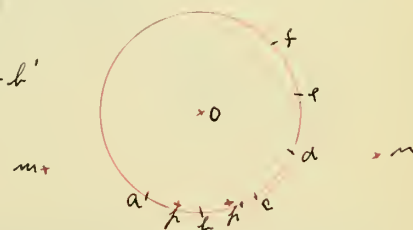
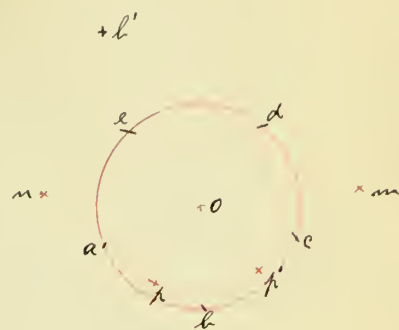
Proof:  $\angle abc = 150^\circ$ .

Const.

Hence  $\angle aob = 30^\circ$ .

(W., Art., 438)

Therefore abcd... is the required dodecagon.





## VIII. A regular pentadecagon.

Let  $a'b'$  be the given side.

To construct a regular pentadecagon having  $a'b'$  as a side.

Construct  $\angle mbn = 156^\circ$ . (Art., 144)

Lay off chords  $ab = bc = a'b'$ . (Art., 4, Cor.)

Construct perpendiculars,  $po$  and  $p'o$ , at the middle points of  $ab$  and  $bc$  and find their intersection  $o$ . (Arts., 12, 7)  
Construct a circle with  $o$  as a centre and  $oa$  as a radius and lay off chords  $ab = bc = \text{etc.}$  Then  $abcd\dots$  is the required pentadecagon!

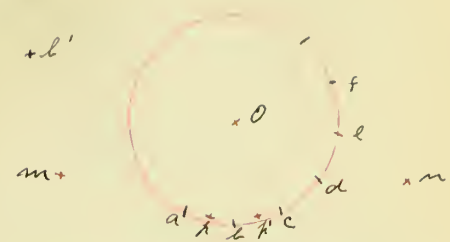
Proof:  $\angle abc = 156^\circ$ .

Const.

Hence  $\angle aob = 24^\circ$ .

(W. Art., 438)

Therefore  $abcd\dots$  is the required pentadecagon.



164. To divide a given trapezoid into two equivalent parts by a line parallel to the bases.

Let  $b''c''cb$  be the given trapezoid.

To divide  $b''c''cb$  into two equivalent parts by a line parallel to  $b''c''$ .

Suppose  $b'c'$  is the required line.

Find the intersection,  $a$ , of  $b''b$  and  $c''c$ .

Construct  $ad''$  perpendicular to  $b''c''$

and find its intersections,  $d$  and  $d'$ , with  $bc$  and  $b'c'$ . (Arts., 12, 7)

$\text{Tri. } abc \approx (ad \times bc) \div 2$  and  $\text{trapezoid } b''c''cb \approx ((b''c'' + bc) \div 2) dd''$ . (W., Arts. 403, 407)

Then  $\text{tri. } ab'c' \approx ((ad + bc) \div 2) + 1/2((b''c'' + bc) \div 2) dd'' \approx (2ad \times bc + (b''c'' + bc) dd'') \div 4$ .

Const.

Triangles  $abc$  and  $ab'c'$  are similar.

(W., Art., 354)

Therefore  $((ad \times bc) \div 2) : (2ad \times bc + (b''c'' + bc) dd'') \div 4 = ad^2 : ad'^2$ . (W., Art., 412)

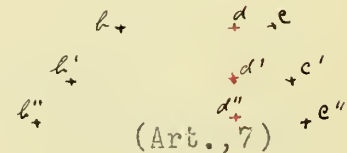
Solving this equation we get  $ad' = \sqrt{ad^2 + (ad(bc + b''c'') dd'') \div 2bc}$ .

Therefore, construct  $ad' = \sqrt{ad^2 + (ad(bc + b''c'') dd'') \div 2bc}$ . (Arts., 5, 8, 9, 10)

Through  $d'$ , construct  $b'c'$  parallel to  $b''c''$ .

(Art., 20)

Then  $b'c'$  is the required line.



165. To divide a given trapezoid into two equivalent parts by a line through a given point in one of the bases.

Let  $abcd$  be the given trapezoid and

$p$  the given point in one of the bases.

To draw a line through  $p$  dividing  $abcd$  into two equivalent parts.



Fig. 1

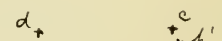
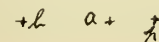
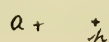


Fig. 2.





Suppose  $pp'$  the line (Fig.1) For brevity let  $dc=m$ ,  $p'c=x$ ,  $ap=n$ ,  $pb=r$ ,  $dp=l$ . Since the two trapezoids  $app'd$  and  $pbcp'$  are equivalent and have the same altitude, we must have  $x+r=m-x+n$ . (W., Art., 407)

Hence  $x=(m+n-r)\div 2$ .

Therefore construct  $cp'=(dc+ap-pb)\div 2$  (Art., 5, 9)

Then  $pp'$  will be the required line.

\*Cor. If  $x$  is negative in this construction one of the equivalent figures will be a triangle. Suppose  $pp'$  the required line. (Fig.2) Let  $p'c'=x$  and the other values be the same as above. Since the triangle  $pp'b$  is equivalent to half the trapezoid  $abcd$  we have

$$((m+n+r)\div 4)l = 1/2rx \text{ or } x=((m+n+r)\div 2r)l$$

Therefore construct  $c'p'=((cd+ap+pb)\div 2pb)dp$  (Arts., 5, 8, 9)

Then  $pp'$  is the required line.

166 To construct a regular pentagon, given one of the diagonals.

Let  $ac$  be the given diagonal.

To construct a regular pentagon with  $ac$  as a diagonal.

Construct  $\angle oac = \angle oca = 18^\circ$ . (Art., 161, Cor.)

With  $o$  as a centre and  $oa$  as a radius, describe a circle.

Inscribe a regular pentagon in this circle. (Art., 143, Cor.)

Then this is the required pentagon.

Proof:  $\angle aoc = 180^\circ - 36^\circ = 144^\circ$ . (W., Art., 129)

Since  $ac$  is a diagonal, the arc  $ac$  must be subtended by two equal chords which are the sides of the required pentagon. Bisect  $\angle aoc$ . (Art., 15)

Then  $\angle aob = 72^\circ = 1/5$  of the right angles. Hence  $abcd$  is the required pentagon.

167. To divide a given straight line into two segments such that their product shall be the maximum.

Let  $ab$  be the given line.

To divide  $ab$  into two segments such that their product shall be the maximum.  $a + c + b$

Bisect  $ab$  at  $c$ . (Art., 14)

Then  $ac$  and  $bc$  are the required segments. (W., Art., 489)

168. To find a point in a semicircumference such that the sum of its distances from the extremities of the diameter shall be





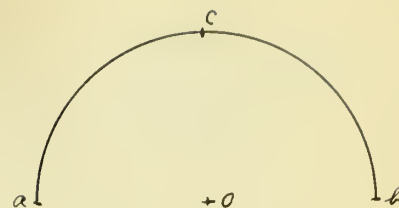
the maximum.

Let  $acb$  be the given semicircumference.

To find a point in  $acb$  such that the sum of its distances from  $a$  and  $b$  shall be the maximum.

Bisect the arc  $acb$  at  $c$ . (Art., 1)

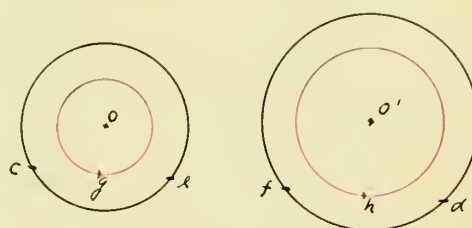
Then  $c$  is the required point. (W., Art., 485)



169. To draw a common secant to two given circles exterior to each other such that the intercepted chords shall have the given lengths  $a, b$ .

Let the outer circles whose centres are  $o$  and  $o'$  be the given circles.

To draw a common secant to  $o$  and  $o'$  such that the intercepted chords shall have the given lengths  $a, b$ .



Construct  $og = \sqrt{oe^2 - a^2/4}$

(Arts., 5, 8, 10)

Then construct a circle with  $o$  as a centre and  $og$  as a radius.

Construct  $o'h = \sqrt{o'f^2 - b^2/4}$ ,

(Arts., 5, 8, 10)

and then construct a circle with  $o'$  as a centre and  $o'h$  as a radius.

Construct a common tangent to the two inner circles whose centres are  $o$  and  $o'$ .

(Art., 64)

Then this is the required secant.

Proof:  $cg = ge$  and  $fh = hd$ .

(W., Art., 245)

$ce = a$  and  $fd = b$ .

(W., Art., 249)

Therefore  $cd$  is the required secant.

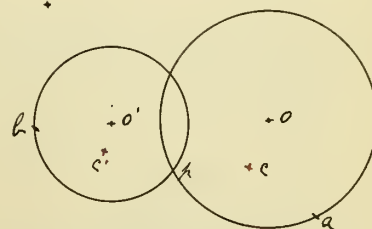
170. To draw through one of the points of intersection of two intersecting circles a common secant which shall be of given length.

Let  $o$  and  $o'$  be the given circles,  $p$  their point of intersection and  $m$  the length of the required secant.

To draw a common secant through  $p$  having the given length  $m$ .

Construct the right triangle  $oo'c$  having hypotenuse  $oo'$  and leg  $o'c = m/2$

Construct  $ab$  parallel to  $o'c$  and through the point  $p$ . (Art., 20)





Then  $ab$  is the required secant.

Proof:  $oc$  and  $o'e'$  are perpendicular to  $ab$ .

Const.

Hence  $oc$  bisects  $ap$  and  $o'e'$  bisects  $pb$ .

(W., Art., 245)

But  $o'e' = m/2$ .

Const.

Therefore  $ab$  is the required secant.

171. To construct an isosceles triangle given the altitude and one of the equal base angles.

Let  $\phi$  be the given angle and  $p$  the given altitude.

To construct an isosceles triangle with altitude  $p$  and  $\phi$  as one of the equal angles.

On any line  $mn$  construct  $cd$  perpendicular to  $mn$ .

(Art., 12)

Lay it off equal to  $p$ .

(Art., 4, Cor.)

Construct  $\angle dca = \angle dc b = 180^\circ - (\phi + 90^\circ) = 90^\circ - \phi$

and find the points,  $a$  and  $b$ , where the lines  $ca$  and  $cb$  intersect  $mn$ .

(Arts., 19, 7)

Then  $abc$  is the required triangle.

Proof:  $cd = p$  and  $\angle cad = \angle cbd = \phi$ .

Const.

Therefore  $abc$  is an isosceles triangle.

(W., Art., 120)

Therefore  $abc$  is the required triangle.

172. To construct an equilateral triangle, given the altitude.

Let  $p$  be the given altitude.

To construct an equilateral triangle having  $p$  as the altitude.

On any line  $mn$ , construct  $cd$  perpendicular to  $mn$ .

(Art., 12)

Lay it off equal to  $p$ .

(Art., 4, Cor.)

Construct  $\angle dca = \angle dc b = 30^\circ$

and find the points,  $a$  and  $b$ , where the lines  $ca$  and  $cb$  intersect  $mn$ .

(Arts., 18, 7)

Then  $abc$  is the required triangle.

Proof:  $cd = p$ .

Const.

$\text{tri. } adc = \text{tri. } bdc$ .

(W., Art., 139)

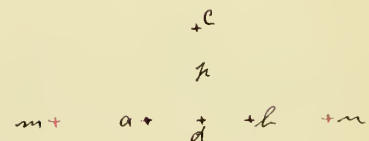
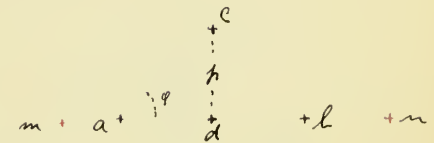
Hence  $\angle cad = \angle cbd$ .

(W., Art., 128)

But  $\angle acb = 60^\circ$ .

Const.

Hence  $\angle acb = \angle cab = \angle cba$ ,





Then triangle abc is equilateral.

(W., Art., 148)

Therefore abc is the required triangle.

173. To construct a right triangle, given the radius of the inscribed circle and the difference of the acute angles.

Let  $m$  be the radius of the inscribed circle and  $\phi$  the difference of the acute angles.

To construct a right triangle whose inscribed circle shall have a radius  $m$  and the difference of whose acute angles is  $\phi$ .

Construct  $ab$  perpendicular to  $bc$ .

Construct the bisector,  $bd$ , of the angle  $abc$ .

At a distance  $m$  from  $bc$ , construct  $m'n'$  parallel to  $bc$ .

Find the intersection  $o$  of  $m'n'$  and  $bd$ .

With  $o$  as a centre and radius  $m$ , describe a circle. This will be tangent to  $ab$  and  $bc$ .

Construct  $\angle pop' = (270^\circ - \phi) \div 2$ .

Construct a tangent to the circle at  $p'$ .

Find its intersections,  $c$  and  $a$ , with  $bc$  and  $ab$  respectively.

Then  $abc$  is the required triangle.

Proof:  $\angle abc = 90^\circ$  and  $pop' = (270^\circ - \phi) \div 2$ .

Hence  $\angle c = 180^\circ - ((270^\circ - \phi) \div 2) = (90^\circ + \phi) \div 2$ .

$\angle p'op'' = 270^\circ - ((270^\circ - \phi) \div 2) = (270^\circ + \phi) \div 2$ .

Hence  $\angle a = 180^\circ - ((270^\circ + \phi) \div 2) = (90^\circ - \phi) \div 2$ .

Then  $\angle c - \angle a = ((90^\circ + \phi) \div 2) - ((90^\circ - \phi) \div 2) = \phi$ .

Therefore  $abc$  is the required triangle.

174. To construct an equilateral triangle so that its vertices shall lie in three given parallel lines.

Let  $mn$ ,  $m'n'$ ,  $m''n''$  be the three given lines.

To construct an equilateral triangle having its vertices on the given lines.

Let  $abc$  be the required triangle.

Circumscribe a circle about the triangle  $abc$  and find its intersection,  $d$ , with  $m'n'$ .

Then  $\angle adb = \angle bdc = 60^\circ$ .

Hence, to construct  $abc$  take any point  $d$  in  $m'n'$  and construct  $\angle bda = \angle bdc = 60^\circ$





(17, Cor.)

Find the intersections,  $d$  and  $c$ , of  $mn$  and  $ad$ ,  $m'n'$  and  $dc$ . (Art., 7)

Then construct an equilateral triangle with  $ac$  as one side. (Art., 17)

Then  $abc$  is the required triangle.

175. To draw a line from a given point to a given straight line which shall be to the perpendicular from the given point as  $m:n$ .

Let  $p$  be the given point,  $ab$  the given line and  $m:n$  the given ratio.

To construct a line  $pd$  such that  $m:n=pc:pd$ .

Construct  $pc$  perpendicular to  $ab$ . (Art., 12)

Find the intersection,  $c$ , of  $ab$  and  $pc$ . (Art., 7)

Construct  $pd$  such that  $m:n=pc:pd$ . (Art., 6)

With  $p$  as centre and  $pd$  as a radius strike an arc cutting  $ab$  at  $d$ .

(Art., 4, II)

Then  $pd$  is the required line.

176. To find a point within a given triangle such that the perpendiculars from the point to the three sides shall be as the numbers  $m, n, p$ .

Let  $abc$  be the given triangle and  $m, n, p$ , the given numbers

To find a point within  $abc$  such that its distances from the sides of the triangle shall be as  $m, n$  and  $p$ .

Construct  $m'n'$  parallel to  $cb$  and at a distance  $n$  from it and find their intersection  $p'$ .

Construct  $p'o$  perpendicular to  $ab$

and lay it off equal to the given length  $p$ .

Through  $o$ , construct  $a'b'$  parallel to  $ab$ .

Construct  $r$  and  $s$  such that  $cb':cb=m:r$  and  $ca':ca=n:s$ . (Art., 6)

Construct lines parallel to  $cb$  and  $ca$  and at distances  $r$  and  $s$  from them and find their intersection  $p''$ .

Let the distance from  $p''$  to  $ab$  be  $t$ . Then  $a'b':ab=p:t$ . (W., Art., 352)

Hence  $p''$  is the required point.

Proof: Tri.  $abc$  and  $a'b'c$  are similar.

Hence  $cb':cb=ca':ca=a'b':ab$ .

Therefore  $m:r=n:s=p:t$  or  $m:n=p:r=s:t$ .



Therefore  $p''$  is the required point.

177. To draw a straight line equidistant from three given points.

Let  $a, b$  and  $c$  be the three given points.

To draw a line equidistant from  $a, b$  and  $c$ .

Construct  $cd$  perpendicular to  $ab$ , (Art., 12, II) and find its intersection  $d$ , with  $ab$ . (Art., 7)

Construct  $a'b'$  perpendicular to  $cd$  at its middle point.

(Art., 12)

Then  $a'b'$  is the required line.

Proof:  $a'b'$  is parallel to  $ab$ .

(W., Art., 104)

Therefore the distance of any point in  $ab$  from  $a'b'$  is the same.

(W., Art., 181)

Every point in  $a'b'$  is equidistant from  $c$  and  $d$ . (W., Art., 160)

Therefore  $c, a$  and  $b$  are equidistant from  $a'b'$ . Hence  $a'b'$  is the required line.

178. To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given straight line shall have a given length.

Let  $o$  be the given circle,  $mn$  the given straight line and  $b$  the given length.

To construct a tangent to  $o$  such that the part included between  $mn$  and the point of contact shall have the length  $b$ .

Let  $r$  be the radius of the given circle and construct  $R = \sqrt{r^2 + b^2}$ .

With  $o$  as a centre and  $R$  as a radius, describe an arc cutting  $mn$  at  $p$  and  $p'$

(Arts., 5, 8, 10)

(Art., 4, II)

From  $p$  and  $p'$  construct the tangents  $pa, pa'', p'a', p'a'''$ . (Art., 36, II)

Then these are the required tangents.

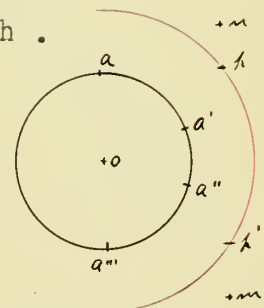
Proof: Triangles  $oap, oa'p'$  etc., are right triangles. (W., Art., 254)

Then  $R^2 - r^2 = ap^2$ .

(W., Art., 372)

Hence  $r^2 + b^2 - r^2 = ap^2$  or  $b^2 = ap^2$  or  $b = ap$ . Similarly for the other tangents.

Therefore these are the required tangents.



178. To inscribe a straight line of a given length between two given circumferences and parallel to a given straight line.

Let  $o$  and  $o'$  be the given circles,  $m$  the given length and  $ab$  the given



line.

To inscribe a line between the two circumferences of  $o$  and  $o'$ , parallel to  $ab$  and equal to  $m$ .

Construct  $oo''$  parallel to  $ab$ .

(Art., 20)

Lay it off equal to  $m$ . (Art., 4, Cor.)

With  $o''$  as a centre describe a circle

having the same radius as  $o$ . Let  $d$  and  $d'$  be the points where this circumference cuts the circumference of  $o'$ .

Construct  $dc$  and  $d'c'$  parallel to  $ab$ .

(Art., 20)

Then  $dc$  and  $d'c'$  are the required lines.

Proof: Since the distance between the centres  $o, o''$  of the two equal circles is  $m$ , then the parallel distance between any two points on the circumferences is  $m$ . Then  $dc = d'c' = m$ .

Therefore  $dc$  and  $d'c'$  are the required lines.

179. To draw through a given point a straight line so that its distances from two other given points shall be in a given ratio.

Let  $a, b$  and  $c$  be the given points and  $m:n$  the given ratio.

To draw a line through  $c$  such that its distances from  $a$  and  $b$  shall be in the ratio  $m:n$ .

Divide  $ab$  at  $e$  in the ratio  $m:n$ . (Art., 85)

Extend  $ce$  to  $d$ .

(Art., 4, Cor1)

Then  $cd$  is the required line.

Proof: Construct  $ap$  and  $bp'$  perpendicular to  $cd$ .

(Art., 12, 11)

Then triangles  $ape$  and  $bp'e$  are similar.

(W., Art., 356)

But  $ae:be = m:n$ .

Const.

Hence  $ap:bp' = m:n$ .

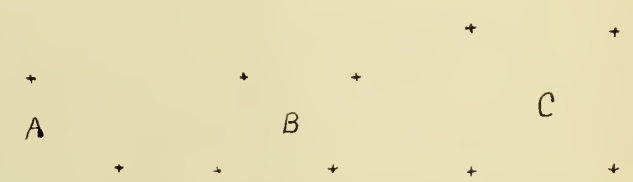
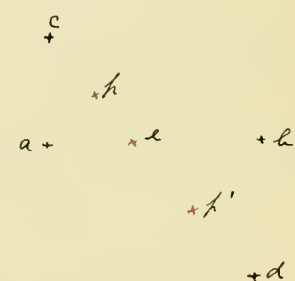
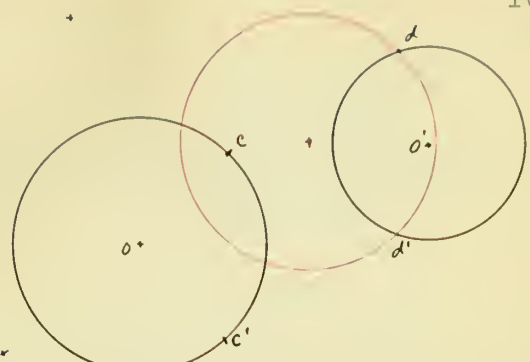
(W., Art., 356)

Therefore  $cd$  is the required line.

180. To construct a square equivalent to the sum of a given triangle and a given parallelogram.

Let  $A$  be the given triangle and  $B$  the given parallelogram.

To construct a square







equivalent to  $A+E$ .

Construct a square equivalent to  $A$ .

(Art., 119, Cor.)

Construct a square equivalent to  $B$ .

(Art., 119)

Construct a square,  $C$ , equivalent to the sum of the two squares already constructed.

(Art., 115)

Then  $C$  is the required square.

181. To construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.

Let  $A$  be the given triangle,  $B$  the given pentagon and  $m$  the difference between the base and altitude of the required rectangle.



To construct a rectangle equivalent to  $A+B$ , the difference of whose base and altitude is  $m$ .

Construct a square  $A'$  equivalent to  $A$ .

(Art., 119, Cor I)

Construct a triangle  $B'$  equivalent to  $B$ .

(Art., 118)

Then construct a square  $B''$  equivalent to  $B'$ .

(Art., 119, Cor. I)

Construct a square  $D$  equivalent to  $(A'+B'')$ .

(Art., 115)

Finally construct a rectangle  $C'$  equivalent to the square  $D$  and having the difference of its base and altitude equal to  $m$ .

(Art., 121)

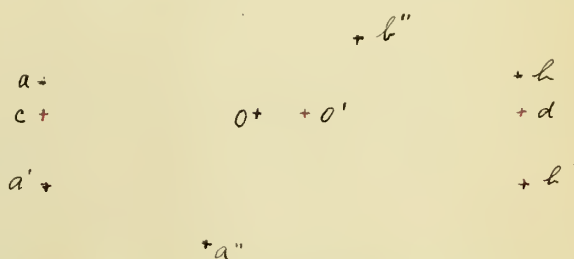
Then  $C'$  is the required rectangle.

182. To construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.

This is a special case of Art., 122. Both of the given polygons must be reduced to squares and then the required polygon can be constructed.

183. To find a point whose distances from three given straight lines shall be as the numbers  $m, n, p$ .

Let  $ab, a'b'$  and  $a''b''$  be the three given straight lines and  $m, n, p$  the given numbers.



To find a point whose distances from  $ab, a'b'$  and  $a''b''$  shall be as the



numbers  $m, n, p$ .

Locate  $c$  such that  $ac:a'c=m:n$ . (Art., 85)  
 Construct  $cd$  parallel to  $ab$ . (Art., 20)  
 Find the intersection,  $o'$ , of  $cd$  and  $a''b''$ . (Art., 7)  
 Lay off  $o'o$  such that  $n:p=ca':o'$ ,. (Art., 6)  
 Then  $o$  is the required point.

Proof:  $ac:a'c=m:n$  and  $a'c:o'o=n:p$ . Const.  
 Multiplying one equation by the other we get  $ac:o'o=m:p$ -  
 Hence  $ac:a'c:o'o=m:n:p$ .  
 Therefore  $o$  is the required point.

\*Cor.I. If the three straight lines form a triangle this exercise reduces to Art., 176.

\*Cor.II. If the three lines are parallel the solution is impossible except for particular values of  $m, n$  and  $p$ .

184. Given an angle and two points  $P$  and  $P'$  between the sides of the angle. To find the shortest path from  $P$  to  $P'$  that shall touch both sides of the angle.

Let  $abc$  be the given angle and  $P$  and  $P'$  the given points.

To find the shortest path from  $P$  to  $P'$  that shall touch both  $ab$  and  $bc$ .

Construct  $Po$  perpendicular to  $bc$  and  $P'o'$  perpendicular to  $ab$ . (Art., 12, II)  
 Extend  $Po$  to  $p$  and  $P'o'$  to  $p'$  making  $P'o'=o'p'$  and  $Po=op$ . (Art., 4, Cor.)

Find the intersections,  $m$  and  $m'$ , of  $pp'$  with  $bc$  and  $ab$  respectively. (Art., 7)

Then  $Pmm'P'$  is the required path.

Proof:  $Pm=pm$  and  $P'm'=p'm'$ . (W., Art., 160)  
 Hence  $Pmm'P'=pp'$ . If we should take any other points,  $n$  and  $n'$ , then  $Pnn'P'=pnn'p'$ . But  $pnn'p'$  is greater than  $pmm'p'$  or  $pp'$ . (W., Art., 49)  
 Hence  $Pnn'P'$  is greater than  $pp'$  or  $Pnn'P'$  is greater than  $Pmm'P'$ .  
 Therefore  $Pmm'P'$  is the required path.

185. To construct a triangle, given its angles and its area.

Let  $A$  be the given area and  $bac, abc, acb$  the given angles.

To construct a triangle with area  $A$  and

$$\begin{array}{ccccccc}
 & & & +c' & +m & & \\
 & & & & & & \\
 & & +c & & & & \\
 & & +h & & +h' & +r & \\
 & & +h & & +h' & & \\
 & & & & & +m & 
 \end{array}$$



angles  $bac, abc$  and  $acb$ .

Construct any triangle,  $abc$ , having the given angles. (Art., 19)  
 Construct  $ap$  perpendicular to  $bc$ . (Art., 12, II)  
 Extend  $ac$  to  $m$ ,  $ab$  to  $n$  and  $ap$  to  $r$ . (Art., 4, Cor.)  
 Construct  $ap' = \sqrt{(2A \times ap) + bc}$  (Arts., 8, 9, 10)  
 Through  $p'$  construct  $b'c'$  parallel to  $bc$   
 and find its intersections,  $c'$  and  $b'$ , with  $am$  and  $an$ . (Arts., 20, 7)  
 Then  $ab'c'$  is the required triangle.

Proof: Triangles  $abc$  and  $ab'c'$  are similar. (W., Art., 354)  
 Hence  $\text{tri. } abc : \text{tri. } ab'c' = \overline{ap}^2 : \overline{ap'}^2$ . (W., Art., 412)  
 But  $\text{tri. } abc = 1/2 ap \times bc$ , (W., Art., 403)  
 and  $\overline{ap'}^2 = (2A \times ap) + bc$ . Const.  
 Then  $\text{tri. } ab'c' \times \overline{ap}^2 = 1/2 ap \times bc \times ((2A \times ap) + bc)$  or  $\text{tri. } ab'c' = A$ .  
 Therefore  $ab'c'$  is the required triangle.

186. To transform a given triangle into a triangle similar to another given triangle.

This is a special case of Art., 122. The triangles are reduced by Art., 119, Cor. I.

187. Given three points  $A, B, C$ . To find a fourth point  $P$  such that the areas of the triangles  $APB, APC, BPC$  shall be equal.

This is another statement of Art., 138 with the exception of the case where  $A, B$  and  $C$  lie in the same straight line, in which case the solution is impossible.

188. To construct a triangle, given its base, the ratio of the other sides and angle included by them.

Let  $m$  be the given base,  $n:p$  the ratio of the other sides and  $bac$  the included angle.

To construct a triangle with base  $m$ , the other two sides having the ratio  $n:p$  and an included angle  $bac$ .

Lay off  $ab' = n$  and  $ac' = p$ . (Art., 4, Cor.)  
 Construct  $ac$  such that  $b'c' : ac' = m : ac$ . (Art., 6)  
 From  $c$ , construct  $cb$  parallel to  $c'b'$ .  
 and find its intersection,  $b$ , with  $ab$ . (Arts., 20, 7)





Then  $abc$  is the required triangle.

Proof:  $bac$  is the given angle.

Triangles  $abc$  and  $ab'c'$  are similar.

Then  $ab':ac'=ab:ac=n:p$  and  $b'c':ac'=bc:ac$ .

But  $b'c':ac'=m:ac$ .

Hence  $bc=m$ .

Therefore  $abc$  is the required triangle.

Const.

(W., Art., 357)

(W., Art., 351)

Const.

189. To divide a given circle into equivalent parts by concentric circumferences.

Let  $o$  be the given circle and  $r$  its radius.

To divide the circle  $o$  into  $n$  equal parts by concentric circumferences.

Construct  $r'=r\sqrt{1/n}$ ,  $r''=r\sqrt{2/n}$ ,  $r'''=r\sqrt{3/n}$  etc.

(Arts., 8, 9, 10)

Construct circles with  $o$  as a centre and  $r'$ ,  $r''$ ,  $r'''$  as radii.

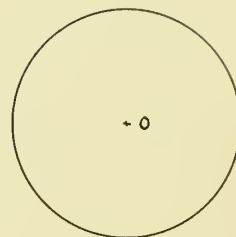
Then these are the required circumferences.

Proof: The areas of these separate circles are  $\pi r'^2 = \pi r^2/n$ ,  $\pi r''^2 = 2\pi r^2/n$ ,  $\pi r'''^2 = 3\pi r^2/n$  etc.

(W., Art., 463)

The difference between the areas of any two of these circles is  $\pi r^2/n$ .

Therefore the original circle is divided into equal parts.



190 In a given equilateral triangle to inscribe three equal circles tangent to each other, each circle tangent to two sides of the triangle.

Let  $abc$  be the given triangle.

To inscribe three equal circles in  $abc$  tangent to each other and each tangent to two sides of  $abc$ .

Bisect angles  $a$ ,  $b$  and  $c$  by  $aa'$ ,  $bb'$ ,  $cc'$ .

(Art., 15)

These intersect in a point  $p$ .

Let  $c'f$  bisect the angle  $bc'c$  and find the intersection,  $o$ , of  $c'f$  and  $bb'$ .

(Art., 7)

Construct  $oe$  perpendicular to  $ab$ .

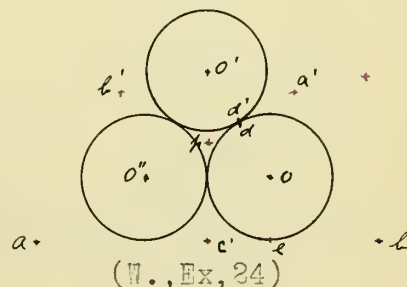
(Art., 12, II)

With  $o$  as a centre and  $oe$  as a radius, describe a circle.

Make  $po'=po''=po$ .

(Art., 4, Cor.)

With  $o'$  and  $o''$  as centres and the same radius as before, describe the other two circles. Then  $o$ ,  $o'$  and  $o''$  are the required circles.





Proof:  $o$  is equidistant from  $bc'$ ,  $c'p$ ,  $pa'$  and  $a'b$ . (W., Art., 162)  
Hence the circles are tangent to the sides of the triangle.  
In the right triangle  $opd$  and  $o'pd'$  (W., Art., 254)  
 $op = o'p$  and  $o'd' = od$ . Const.  
Then triangle  $opd = \text{triangle } o'pd'$ . (W., Art., 151)  
Hence  $pd = pd'$  and the circles are tangent at  $d$ . Similarly the other circles are tangent to each other. Therefore  $o, c'$  and  $o''$  are the required circles..

191. Given an angle and a point  $P$  between the sides of the angle. To draw through  $P$  a straight line that shall form with the sides of the angle a triangle with perimeter equal to a given length  $a$ .

Let  $bcd$  be the given angle,  $P$  the given point and  $a$  the given perimeter.

To draw a line through  $P$  forming with the sides of the given angle a triangle whose perimeter is  $a$ .

Take a line  $mn$  equal to  $a$  and on it construct the segment of a circle in which an angle,  $90^\circ + \frac{1}{2}\angle bcd$ , may be inscribed. (Art., 38)

With any point  $p$ , in the arc  $mpn$  as a centre and a radius  $Pc$ , describe an arc cutting  $mn$  at  $p'$ . (Art., 4, I)

Erect perpendiculars at the middle points of  $mp$  and  $np$  and find their intersections  $m'$  and  $n'$  with  $mn$ . (Arts., 12, 7)

Lay off  $cb' = pm'$ . (Art., 4, Cor.)

Find the intersection  $d'$  of  $cd$  and  $b'P$ . (Art., 7)

Then  $ab'd'$  is the required triangle.

Proof:  $mm' = pm'$  and  $nn' = pn'$ . (W., Art., 160)

Hence the perimeter of triangle  $pm'n' = a$ .

$\angle pmm' = \angle mpm'$  and  $\angle n'pn = \angle pnn'$ . (W., Art., 145)

$\angle mpn = 90^\circ + \frac{1}{2}\angle bcd$ . Const.

$\angle pmm' + pnn' + 90^\circ + \frac{1}{2}\angle bcd = 180^\circ$  or  $\angle pmm' + pnn' + \frac{1}{2}\angle bcd = 90^\circ$ . (W., Art., 129)

$\angle mpn = 90^\circ + \frac{1}{2}\angle bcd = \angle pmm' + \angle pnn' + \angle m'pn'$

or  $\angle pmm' + \angle pnn' = 90^\circ + \frac{1}{2}\angle bcd - \angle m'pn'$ .

Substituting this in the equation above we get

$90^\circ + \frac{1}{2}\angle bcd - \angle m'pn' + \frac{1}{2}\angle bcd = 90^\circ$  or  $\angle m'pn' = \angle bcd$ .

Hence triangle  $m'pn' = \text{triangle } b'cd'$ . (W., Art., 143)

Therefore the perimeter of triangle  $b'cd' = a$  and it is the required triangle.

192. In a given square to inscribe four



equal circles, so that each circle shall be tangent to two of the others and also tangent to two sides of the square.

Let  $abcd$  be the given square.

To inscribe four circles in  $abcd$  each of which shall be tangent to two others and tangent to two sides of the square.

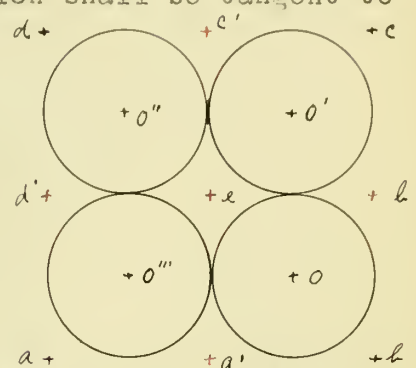
Divide the given square into four equal squares by erecting perpendiculars at the middle points of the sides.

(Art., 12, I)

Find the intersections of the diagonals of the smaller squares. (Art., 7)

With these points as centres and one fourth the side of the original square as radius, describe circles. Then these are the required circles.

Proof: Since the diagonals of a square bisect its angles their point of intersection is equidistant from all four sides. Hence each circle is tangent to the sides of the smaller squares at their middle points. Hence each circle is tangent to two others. Therefore  $o, o', o'', o'''$  are the required circles.



108. In a given square to inscribe four equal circles so that each circle shall be tangent to two of the others and also tangent to one side of the square.

Let  $abcd$  be the given square.

To inscribe four circles in  $abcd$  such that each shall be tangent to two of the others and to one side of the square.

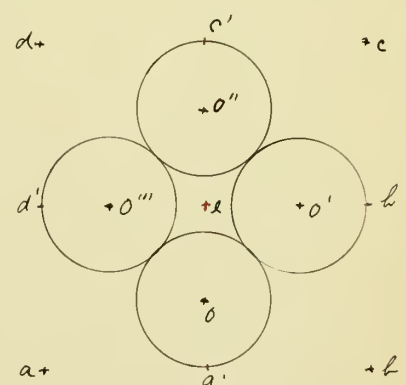
Find the intersection  $e$  of  $ac$  and  $bd$ .

(Art., 7)

Inscribe circles in the triangles  $ade, dec, ceb$  and  $aeb$ .

(Art., 35)

Then these are the required circles.



Proof: Each is tangent to one side of the square. Const.

Since all the triangles are equal, the distance from any vertex of the square to the point of tangency of any of the circles is the same. Hence each circle is tangent to two of the others. Therefore  $o, o', o''$  and  $o'''$  are the required circles.





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